Optimal transport of measures in frequency domain

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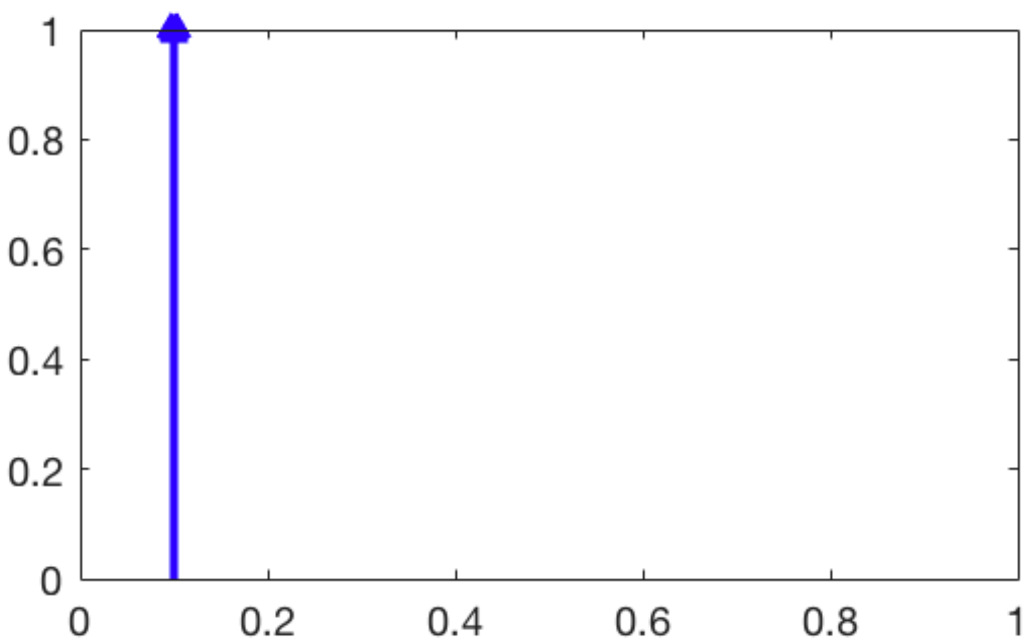
SPARS, Jul. 2019

Context

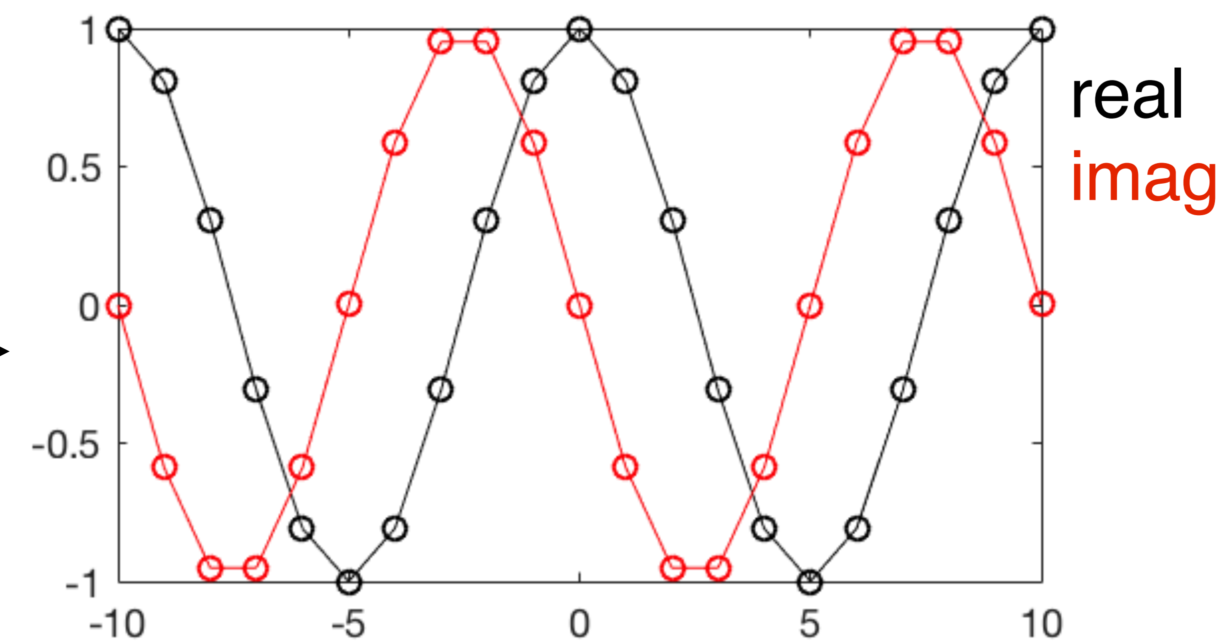
$\mathcal{M}$ = {signed Radon measures on $\mathbb{T} = \mathbb{R} \backslash \mathbb{Z}$}



μ

$\xrightarrow{\mathcal{F}}$

$\mathbb{V} = \{(v_m)_{m=-M}^{M} \in \mathbb{C}^{2M+1}$: $v_{-m} = v_m^*\}$



$$(\mathcal{F}\mu)_m = \int_0^1 e^{-j2\pi fm} d\mu(f)$$

$$m = -M, \dots, M$$

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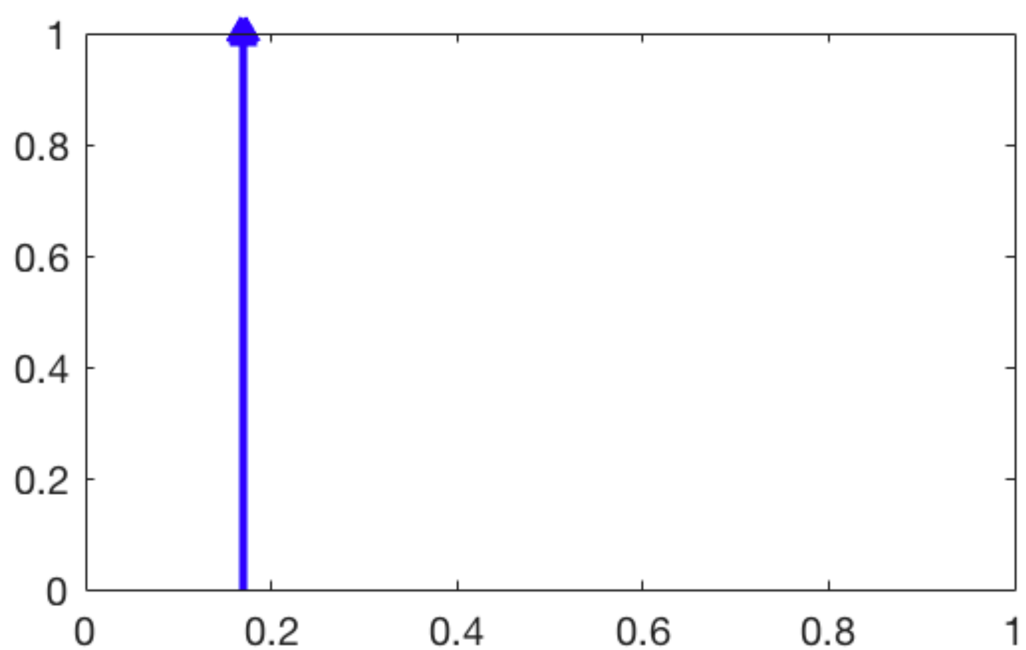

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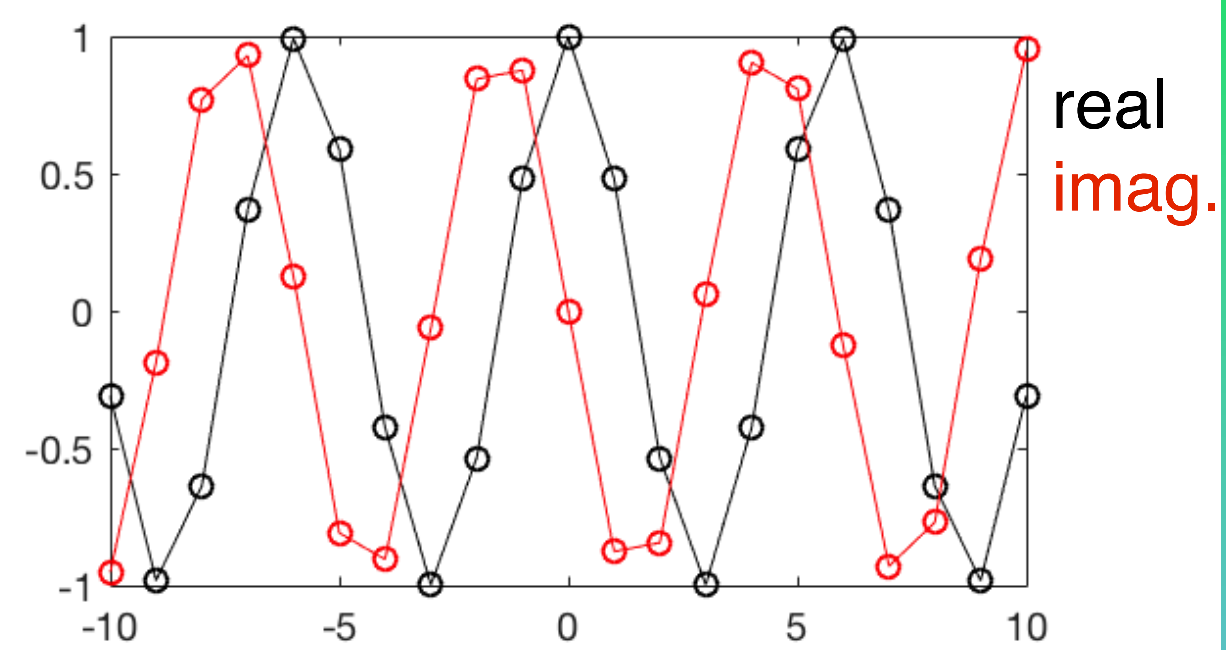

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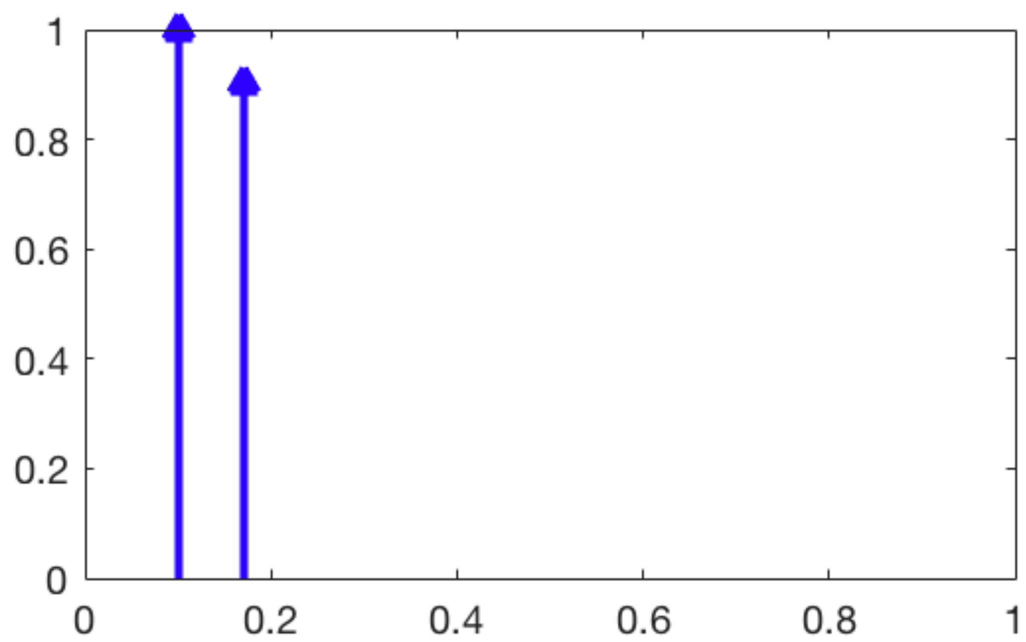

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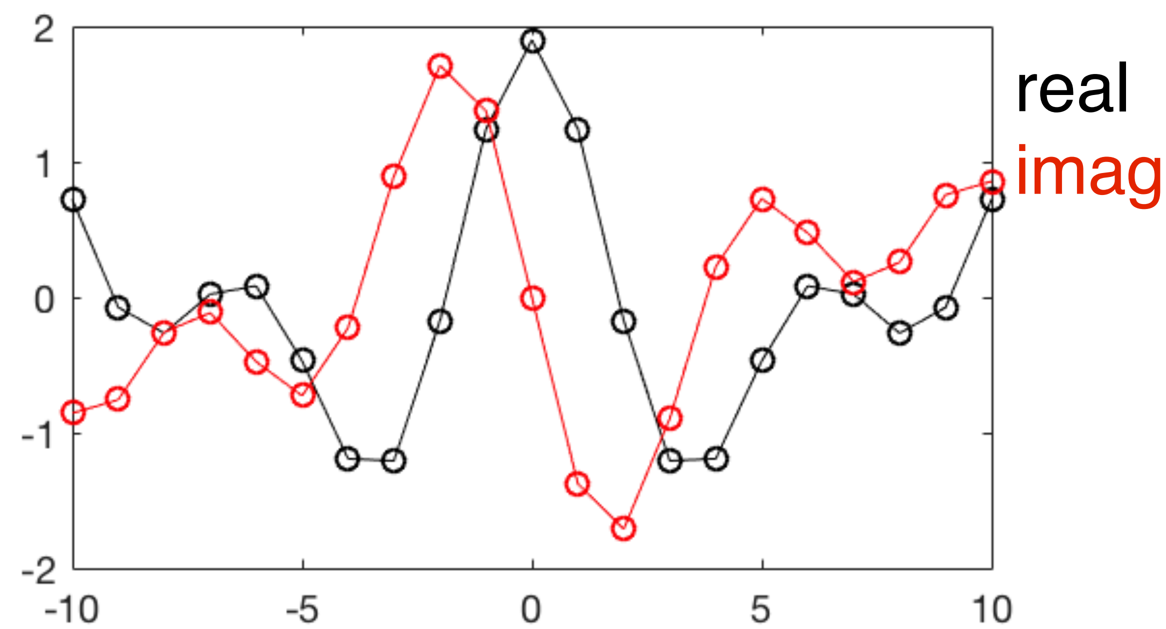

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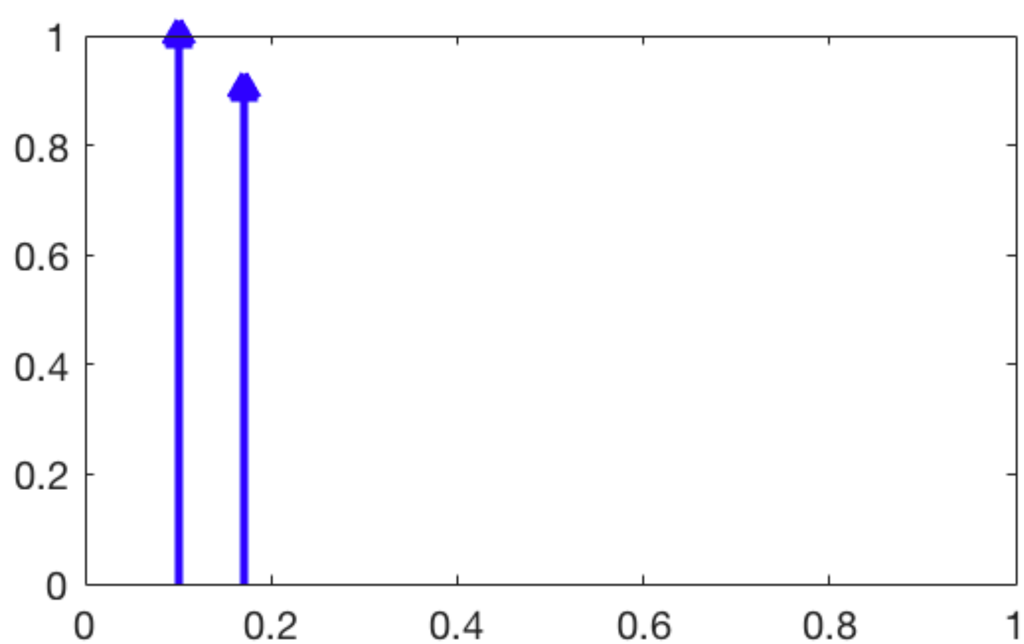
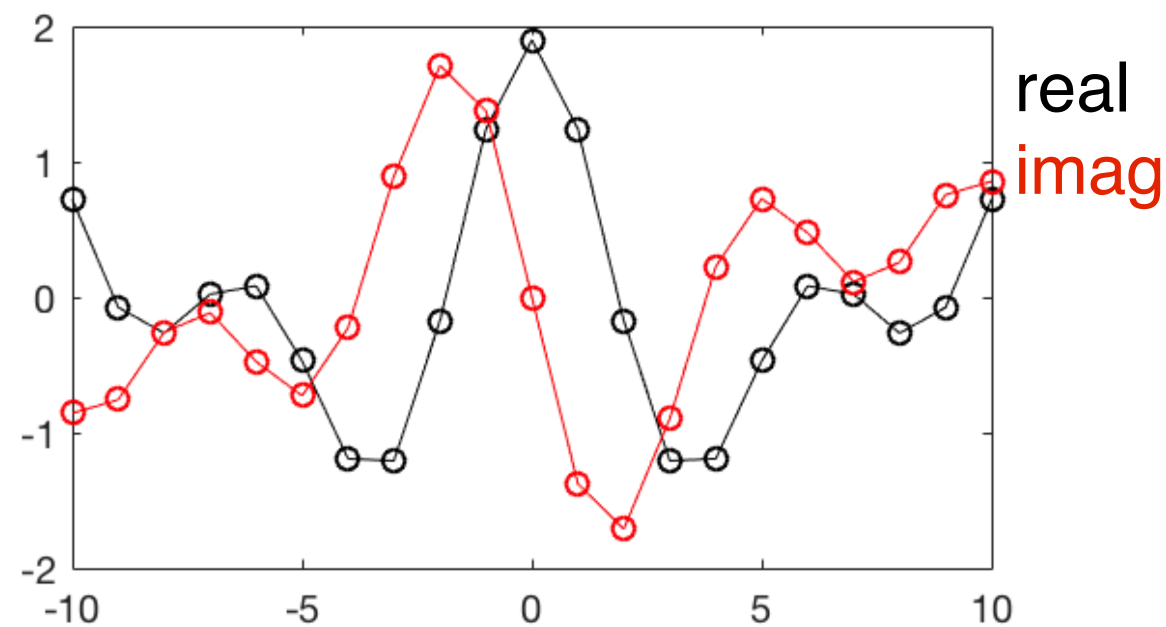


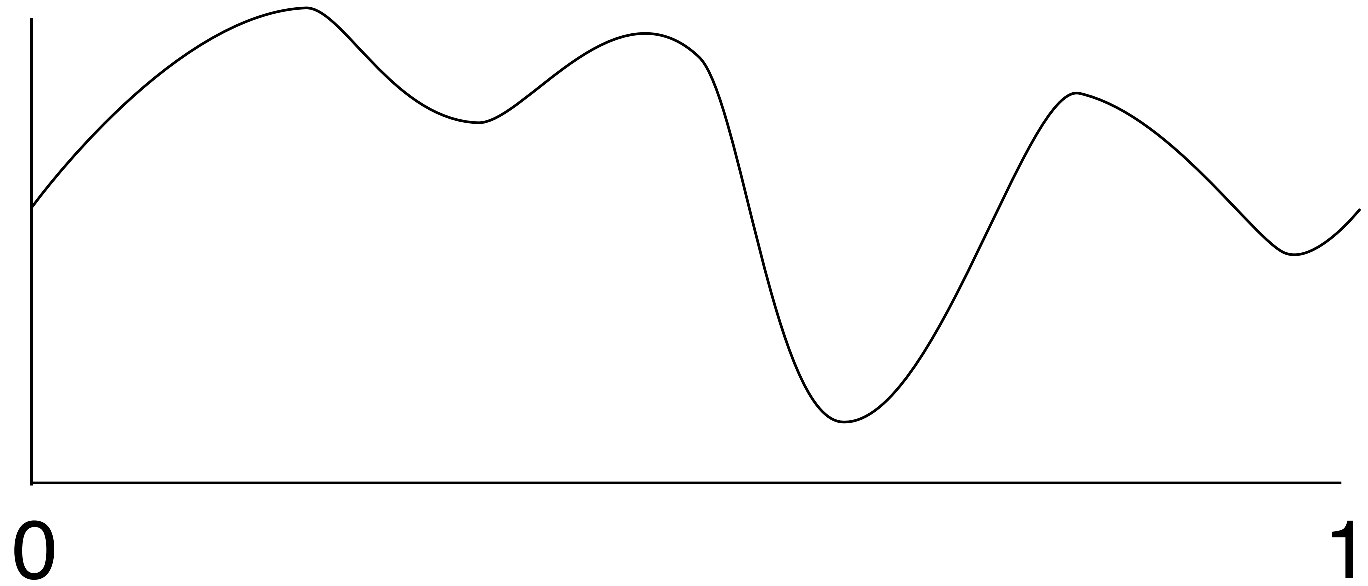
$\xrightarrow{\mathcal{F}}$



About existence, uniqueness, cardinality of μ given $\mathcal{F}\mu$, see

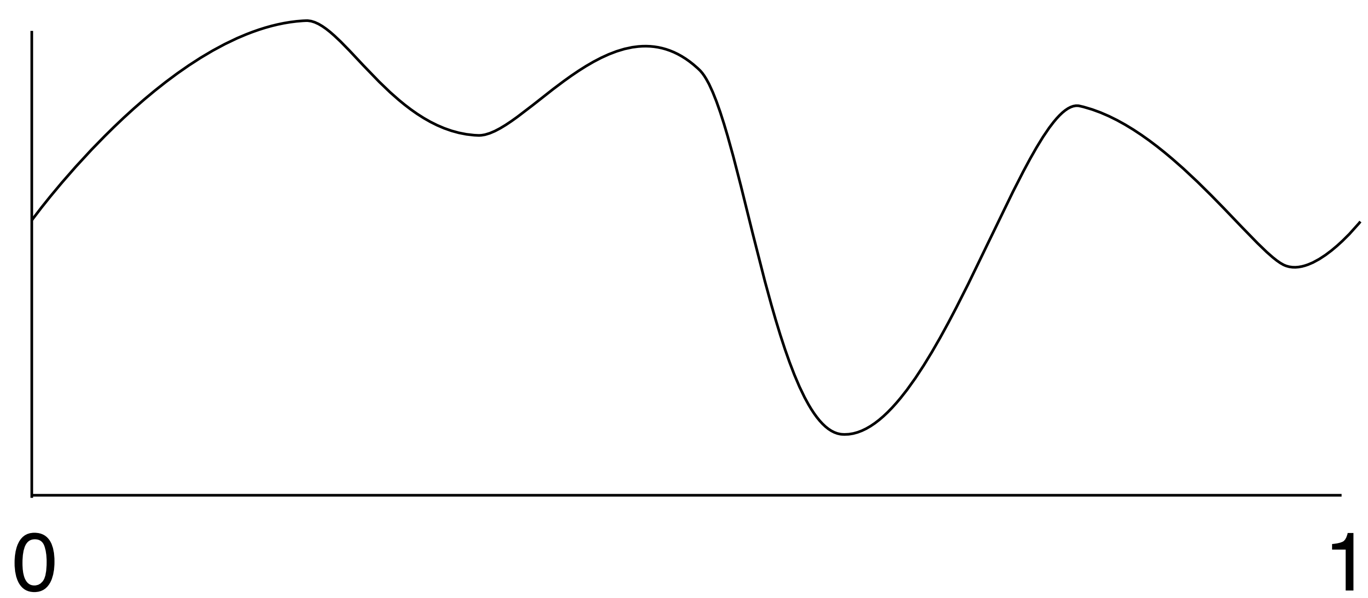
[L. Condat, "Atomic norm minimization for decomposition into complex exponentials," preprint, 2018]

Global optimization



$$\underset{t\in\mathbb{T}}{\text{minimize}}\ f(t)$$

Global optimization



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$$\equiv$$

$$\underset{\text{proba. measure } \mu}{\text{minimize}} \int_{\mathbb{T}} f(t) \mathrm{d}\mu(t)$$

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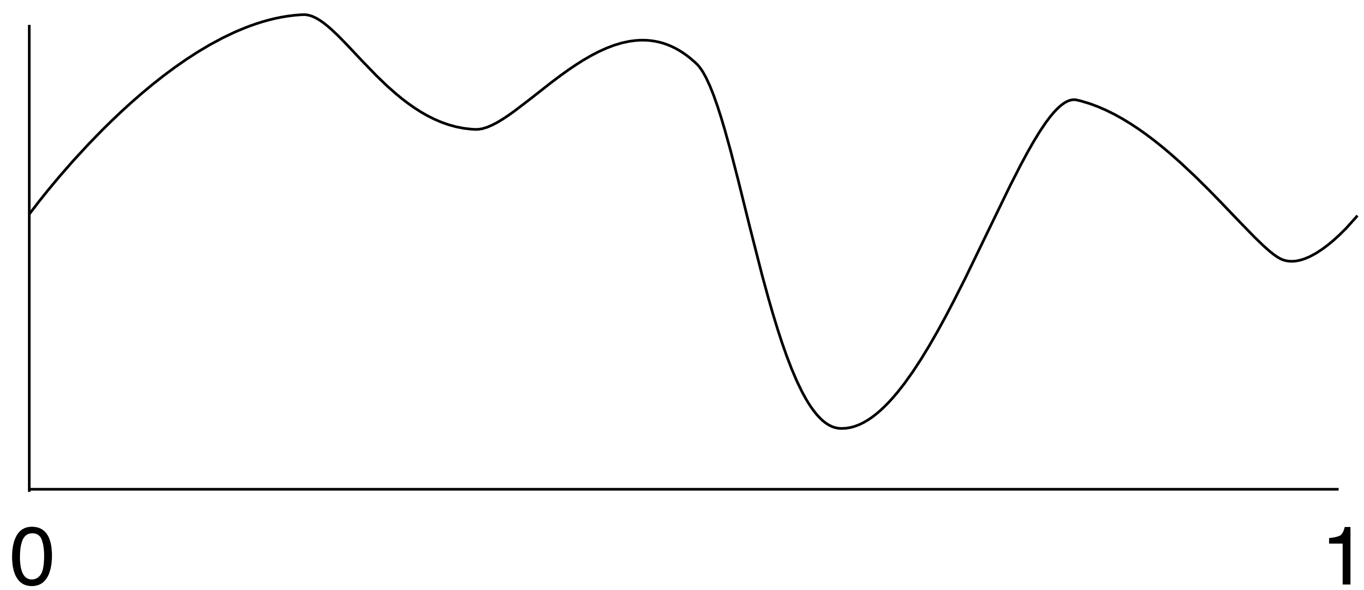

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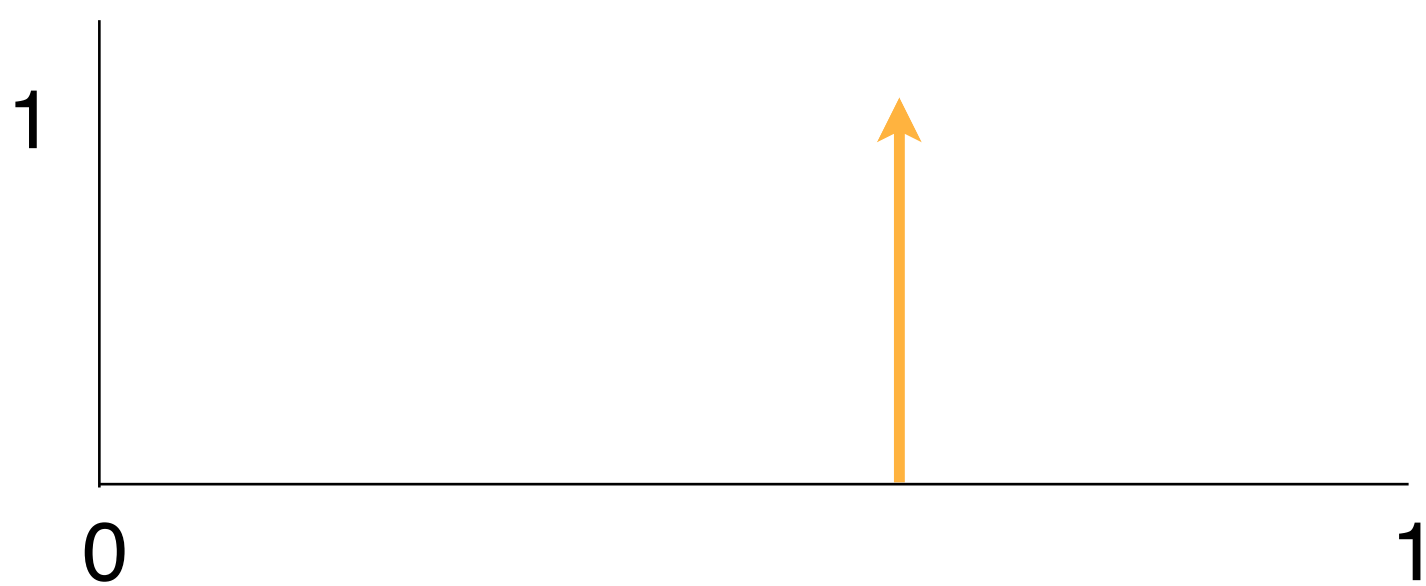

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Global optimization with pairwise costs

$f(t, t') \geq 0$

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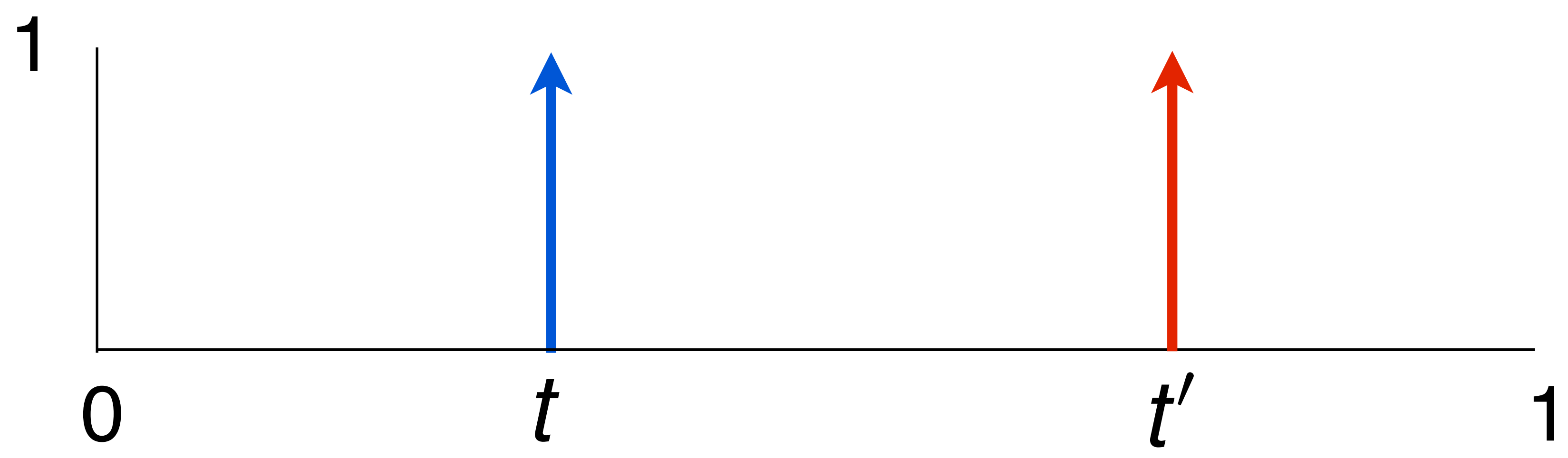
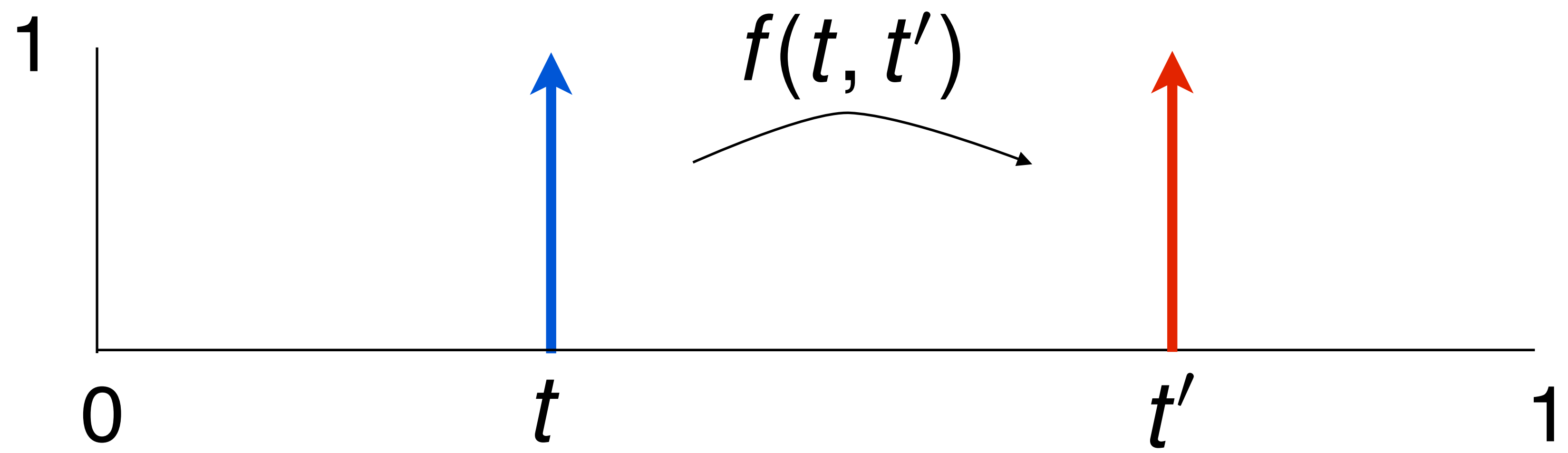


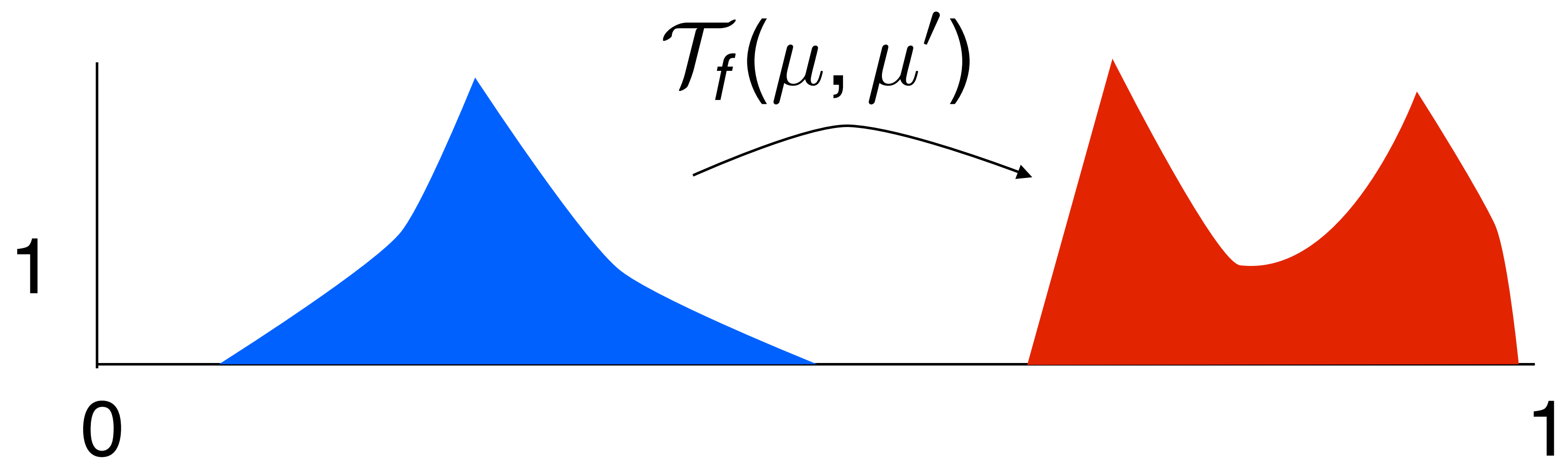
Global optimization with pairwise costs

$f(t, t') \geq 0$: cost of transporting δ_t to $\delta_{t'}$



Optimal transport

Generalization to a pair of positive measures μ and μ' with same mass:



$$\mathcal{T}_f(\mu, \mu') = \inf_{\substack{\text{positive measure} \\ \nu \text{ on } \mathbb{T}^2}} \int_{\mathbb{T}^2} f(t, t')\mathrm{d}\nu(t, t')$$

s.t. the marginals of ν are μ and μ'

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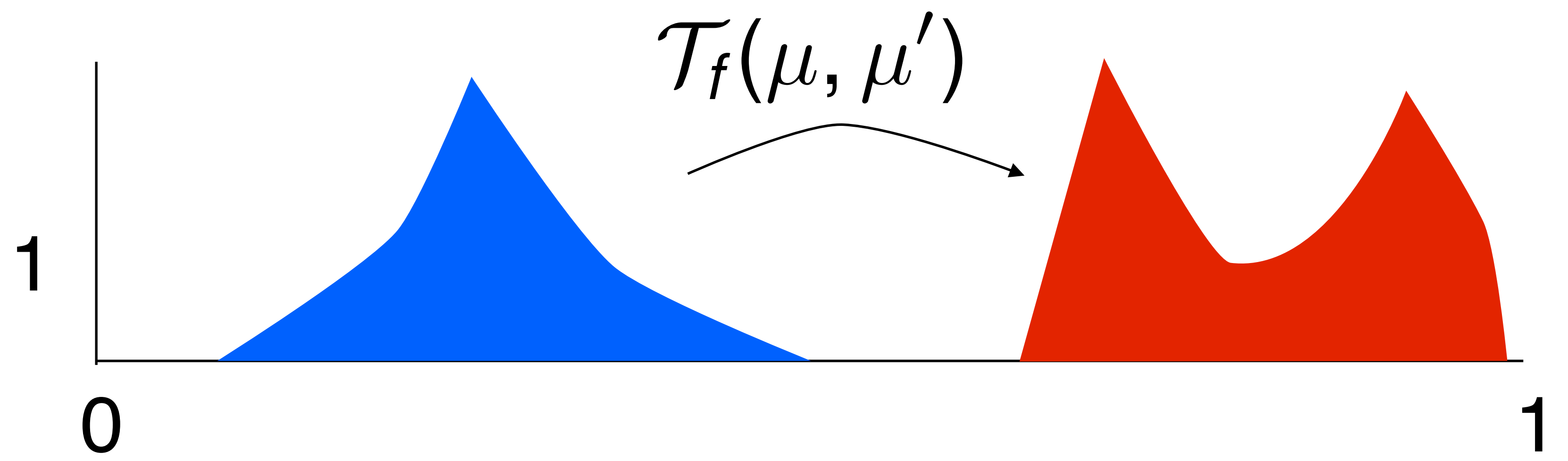

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s.t. the marginals of ν are μ and μ'

This is the largest convex function with $\mathcal{T}_f(c\delta_t, c\delta_{t'}) = cf(t, t')$, for every $c \geq 0$, $(t, t') \in \mathbb{T}^2$

Typical transport costs

$f(t, t') = \{0 \text{ if } t = t', 1 \text{ else}\}$

☞ $\mathcal{T}_f(\mu, \mu') = \frac{1}{2}\|\mu - \mu'\|_{\text{TV}}$ is the Radon distance

---

$f(t, t') = d(t, t')$

☞ $\mathcal{T}_f(\mu, \mu')$ is the 1-Wasserstein distance

$f(t, t') = d(t, t')^2$

☞ $\sqrt{\mathcal{T}_f(\mu, \mu')}$ is the 2-Wasserstein distance

Optimal transport of signed measures

Largest convex function with $\mathcal{T}_f(c\delta_t, c\delta_{t'}) \leq |c| f(t, t')$, for every $c \in \mathbb{R}$, $(t, t') \in \mathbb{T}^2$?

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$\forall (\mu, \mu') \in \mathcal{M}^2$ with $\mu(\mathbb{T}) = \mu'(\mathbb{T})$,

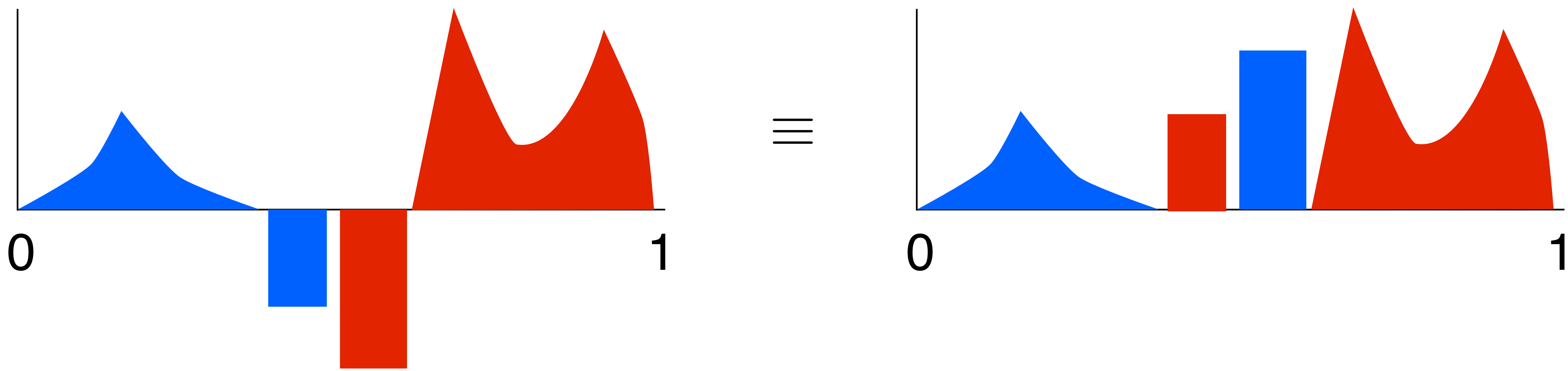
$$\mathcal{T}_f(\mu, \mu') = \inf_{\substack{\text{signed measures} \\ \nu \text{ on } \mathbb{T}^2}} \int_{\mathbb{T}^2} f(t, t') \mathrm{d}|\nu|(t, t')$$

s.t. the marginals of ν are μ and μ'

Optimal transport of signed measures

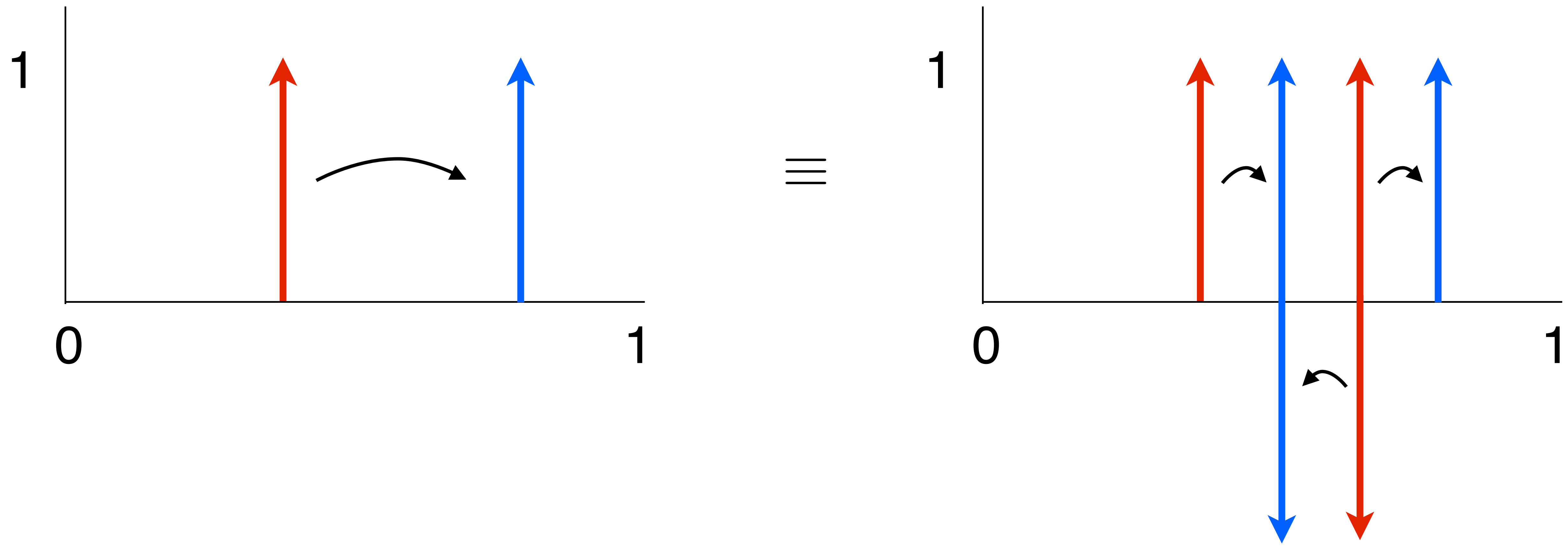
If $f = \phi \circ d$ with ϕ increasing, concave, and $\phi(0) = 0$,

$$\mathcal{T}_f(\mu, \mu') = \mathcal{T}_f(\mu^+ + \mu'^-, \mu'^+ + \mu^-)$$



Optimal transport of signed measures

If $f = \phi \circ d$ with ϕ increasing and strictly convex, $\mathcal{T}_f = 0$!



Atomic norm

For every $v \in \mathbb{V}$, its atomic norm defined as:

$$\|v\|_{\mathrm{a}} = \inf \left\{ \|\mu\|_{\mathrm{TV}} : \ \mu \in \mathcal{M}, \ \mathcal{F}\mu = v \right\}$$

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Finite dimensional SDP formulation:

$$\|v\|_{\mathrm{a}} = \min_{X} \ \frac{2}{M+1}\mathrm{tr}(X) - v_0 \quad \text{s.t.} \quad X \text{ is Toeplitz}$$
$$\text{and} \quad X \succcurlyeq 0 \quad \text{and} \quad X - V \succcurlyeq 0$$

Atomic transport cost

$\forall(v, v') \in \mathbb{V}^2$ with $v_0 = v'_0$, atomic transport cost:

$$\mathcal{T}_{a,f}(v, v') = \min\Big\{\mathcal{T}_f(\mu, \mu') \;:\; (\mu, \mu') \in \mathcal{M}^2,\\ \mathcal{F}\mu = v,\ \mathcal{F}\mu' = v'\Big\}$$

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Limited to the concave case

Atomic Radon distance

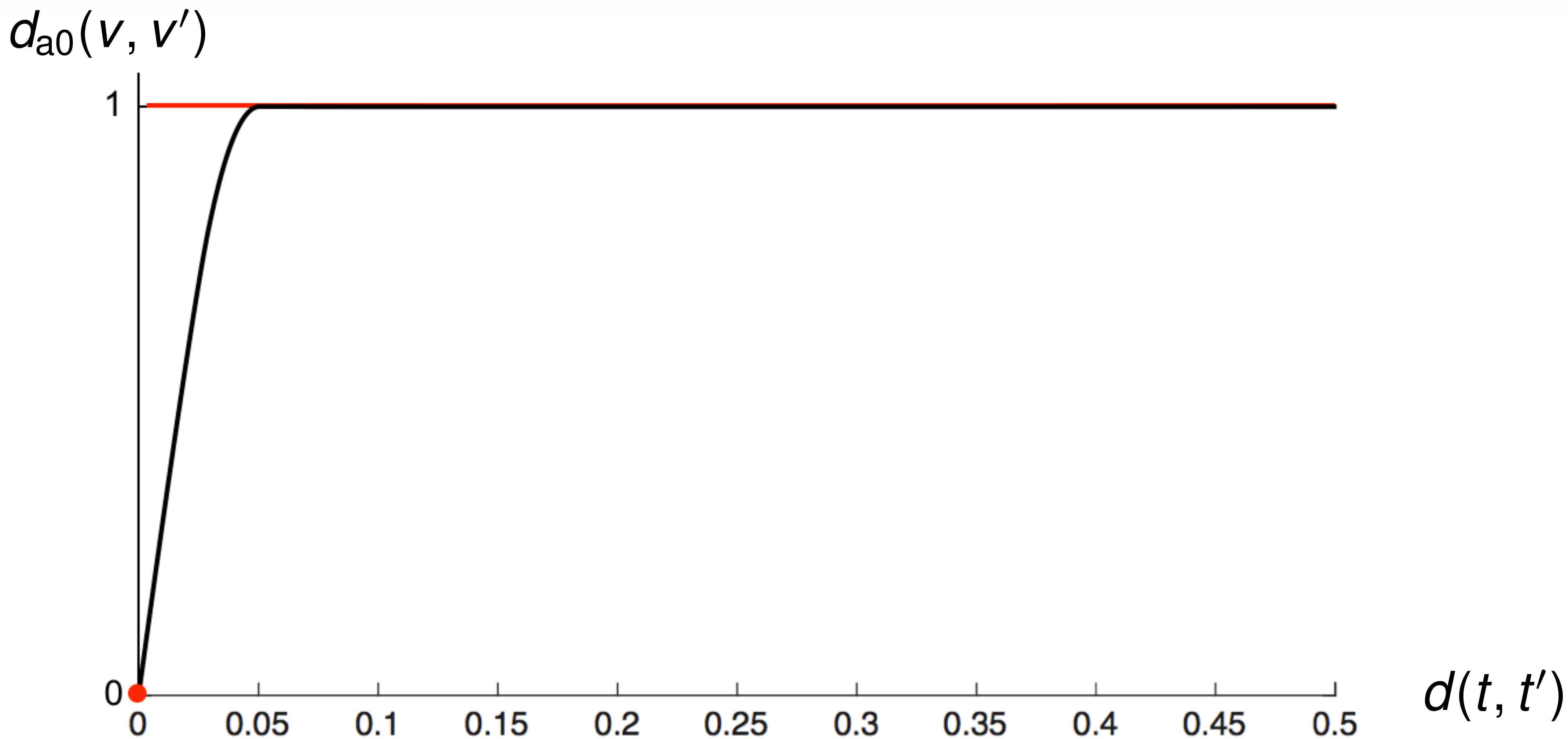
$f(t, t') = \{0 \text{ if } t = t', 1 \text{ else}\}$ ☞ atomic Radon distance

$\forall (v, v') \in \mathbb{V}^2$ with $v_0 = v'_0$,

$$d_{a0}(v, v') = \min \left\{ \tfrac{1}{2} \|\mu - \mu'\|_{TV} : \ (\mu, \mu') \in \mathcal{M}^2, \ \mathcal{F}\mu = v, \ \mathcal{F}\mu' = v' \right\}$$

$$= \tfrac{1}{2} \|v - v'\|_a$$

Atomic Radon distance



$$v = (e^{-j2\pi tm})_{m=-M}^{M},\ v' = (e^{-j2\pi t' m})_{m=-M}^{M},\ M = 10$$

Exact if $d(t, t') \geq \frac{1}{2M}$

Atomic Wasserstein-1 distance

$f(t,t') = d(t,t')$ ☞ atomic Wasserstein-1 distance

$$d_{\mathrm{a1}}(v,v') = \min\Big\{\mathcal{T}_f(\mu,\mu') \;:\; (\mu,\mu')\in\mathcal{M}^2,\\ \mathcal{F}\mu = v,\ \mathcal{F}\mu' = v'\Big\},$$

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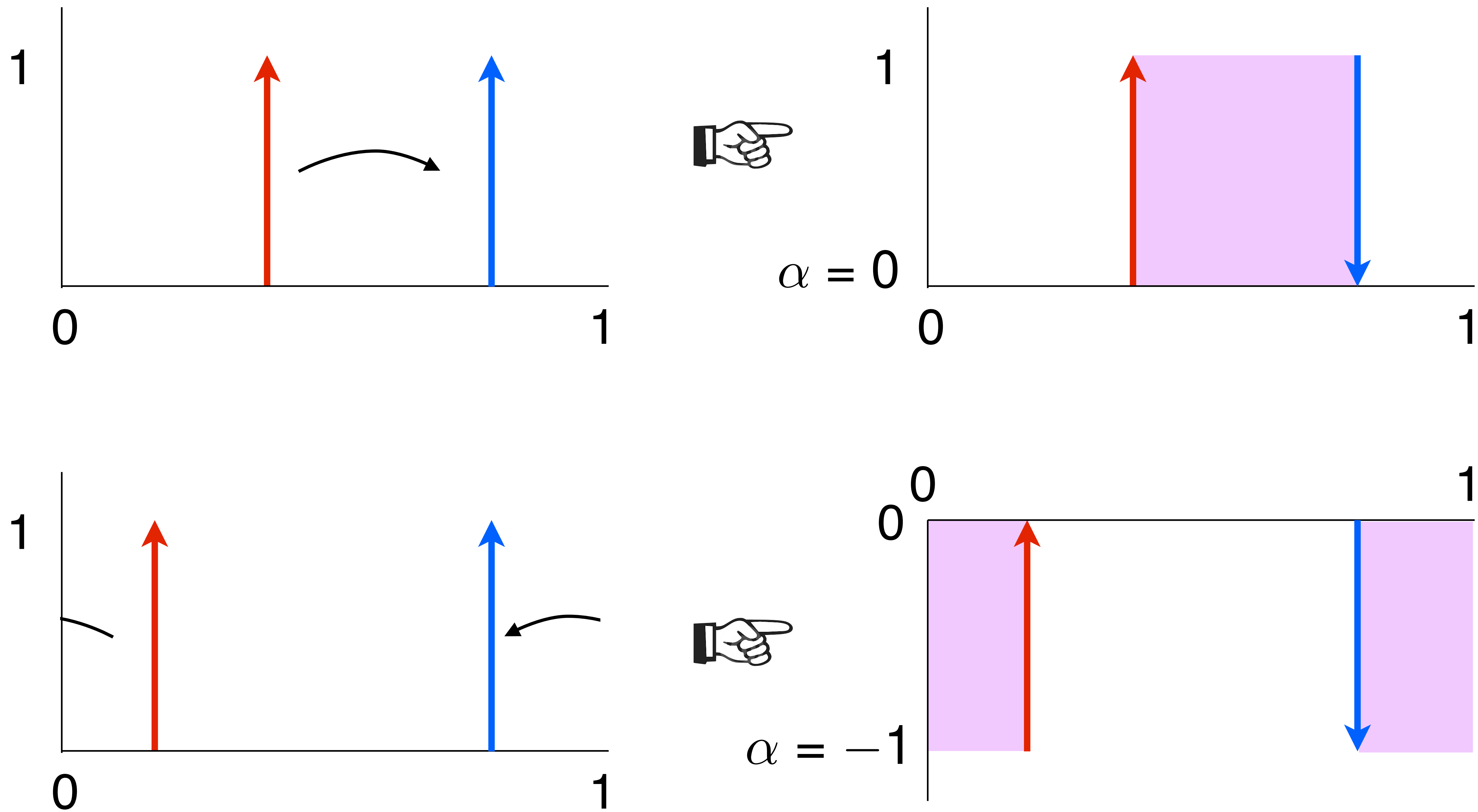
$$\mathcal{T}_f(\mu, \mu') = \min_{\alpha \in \mathbb{R}} \int_{\mathbb{T}} \big| F(t) - F'(t) - \alpha \big| \mathrm{d}t$$

where F and F' are the cumulative functions of μ and μ'

Atomic Wasserstein-1 distance



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$$= \min \Big\{ \|\eta\|_{TV} \;:\; \eta \in \mathcal{M},\; \mathcal{F}\eta = w, \;\text{ with }\; \mathrm{j}2\pi m w_m = v_m - v'_m,\;\; m = -M,\dots,M \Big\}$$

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$$= \min_{X,\alpha} \ \Big(\tfrac{2}{M+1}\mathrm{tr}(X) + \alpha\Big) \quad \text{s.t.} \quad X \text{ is Toeplitz} \quad \text{and} \quad X \succcurlyeq 0 \quad \text{and} \quad X - W + \alpha\mathrm{Id} \succcurlyeq 0$$

where $w = \big((v_m - v'_m)/(\mathrm{j}2\pi m)\big)_{m=-M}^{M}$, with $w_0 = 0$, and $W = \mathrm{T}(w)$

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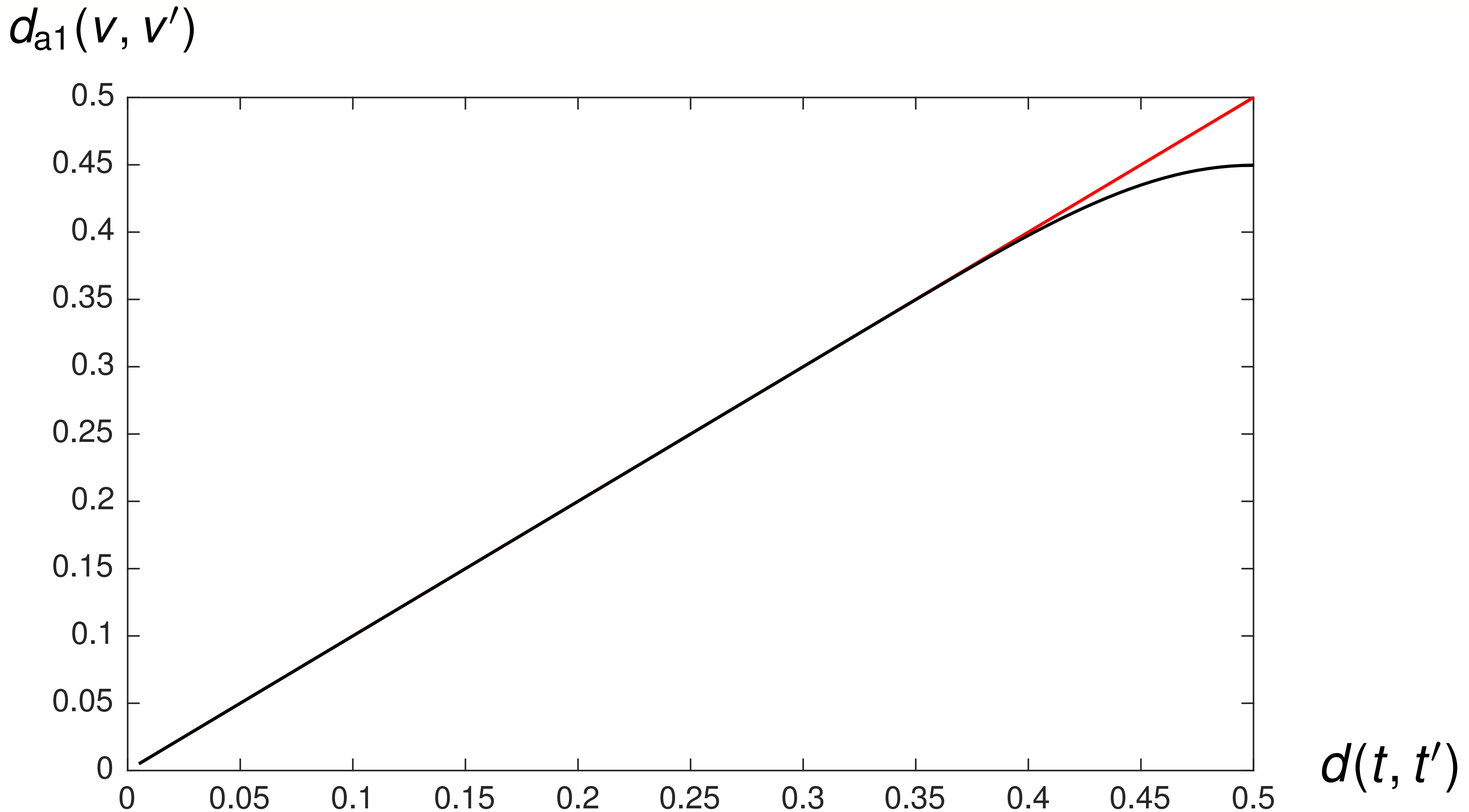
$$= \min_X \Big(\tfrac{2}{M+1}\mathrm{tr}(X) + \mathrm{i}^+(W - X)\Big) \quad \text{s.t.}$$

$$X \text{ is Toeplitz} \quad \text{and} \quad X \succcurlyeq 0,$$

where i^+ denotes the largest eigenvalue

Atomic Wasserstein-1 distance



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Atomic squared Wasserstein-2 distance

v' is fixed as an atom: $v'_m = (c.e^{-\mathrm{j}2\pi t' m})_{m=-M}^{M}$.

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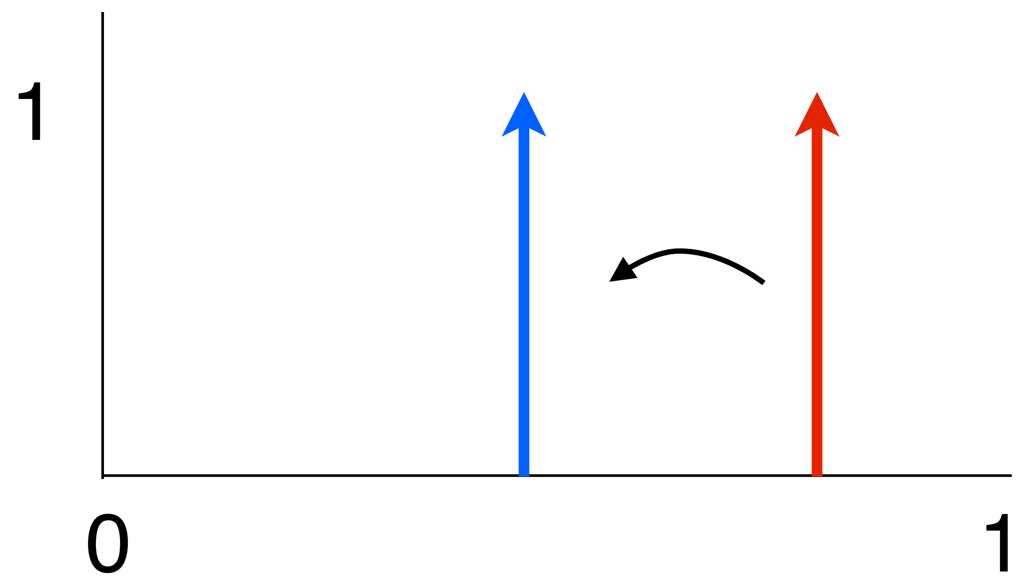
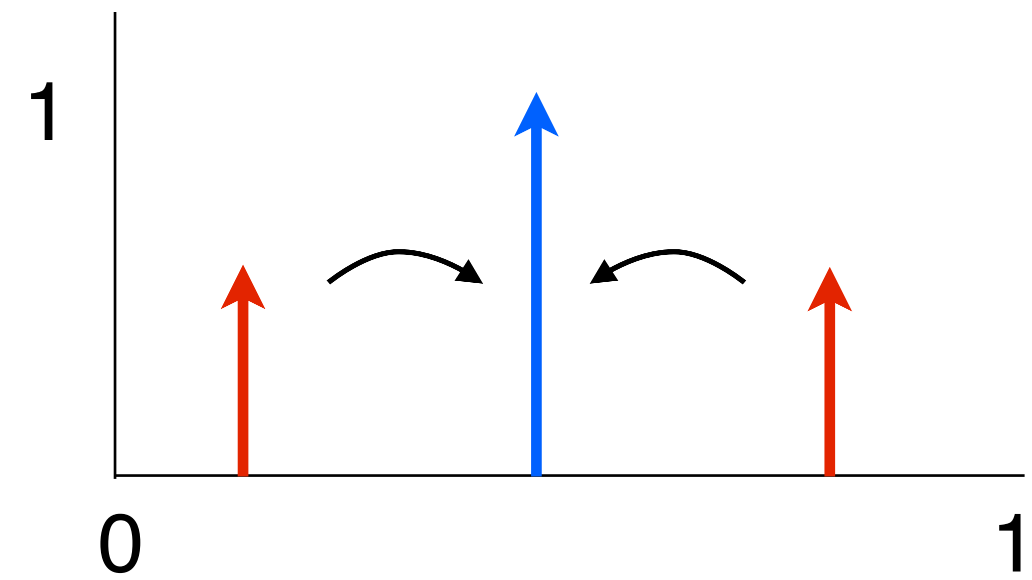
We design an approximation $\widetilde{d}^2_{\mathrm{a2}}$ of the function

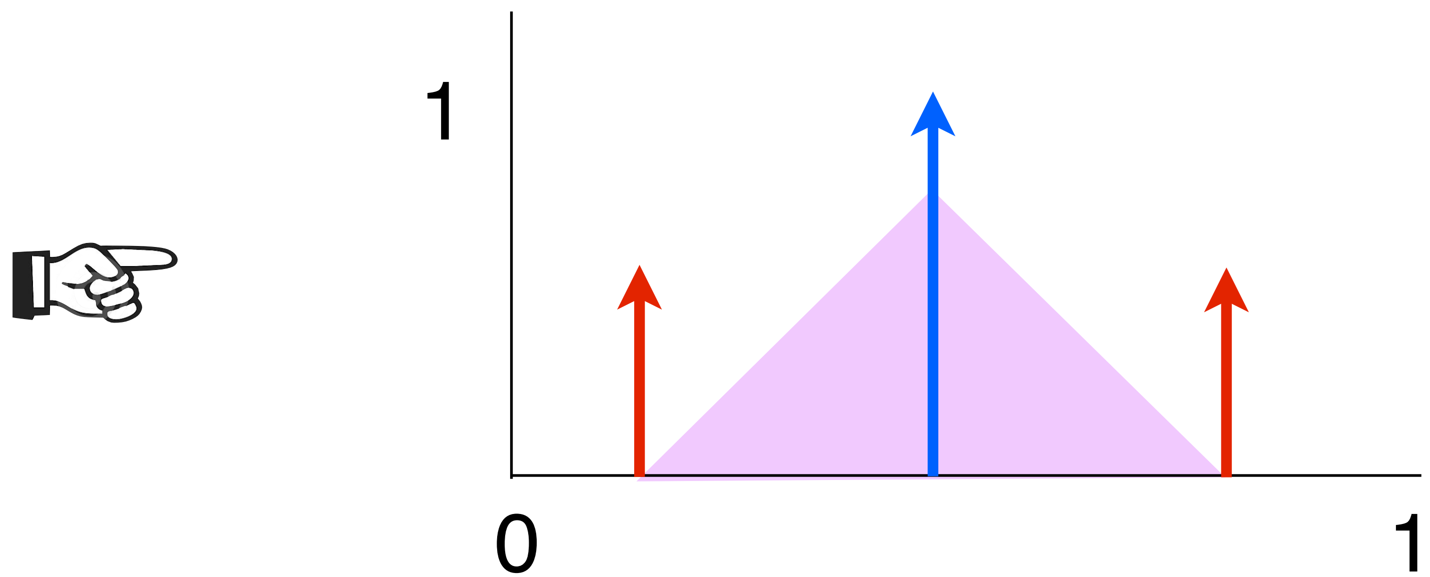
which maps $v \in \mathbb{V}$, with $v_0 = c$ and $\mathrm{T}(v) \succcurlyeq 0$, to

$$\mathcal{T}_{\mathrm{a},d^2}(v, v') = \min_{\text{pos. measure } \mu} \int_{\mathbb{T}} d(t, t')^2 \mathrm{d}\mu(t) \quad \text{s.t. } \mathcal{F}\mu = v$$

Atomic squared Wasserstein-2 distance







Atomic squared Wasserstein-2 distance

$$\widetilde{d}_{\mathrm{a2}}^2(v, v') = \min\Big\{\eta(\mathbb{T}) \;:\; \eta \in \mathcal{M} \text{ is positive}, \; \mathcal{F}\eta = w,$$
$$\text{with } -4\pi^2 m^2 w_m = v_m - 2v'_m + {v'}_m^2 v_m^*, \; m = -M, \dots, M\Big\}$$

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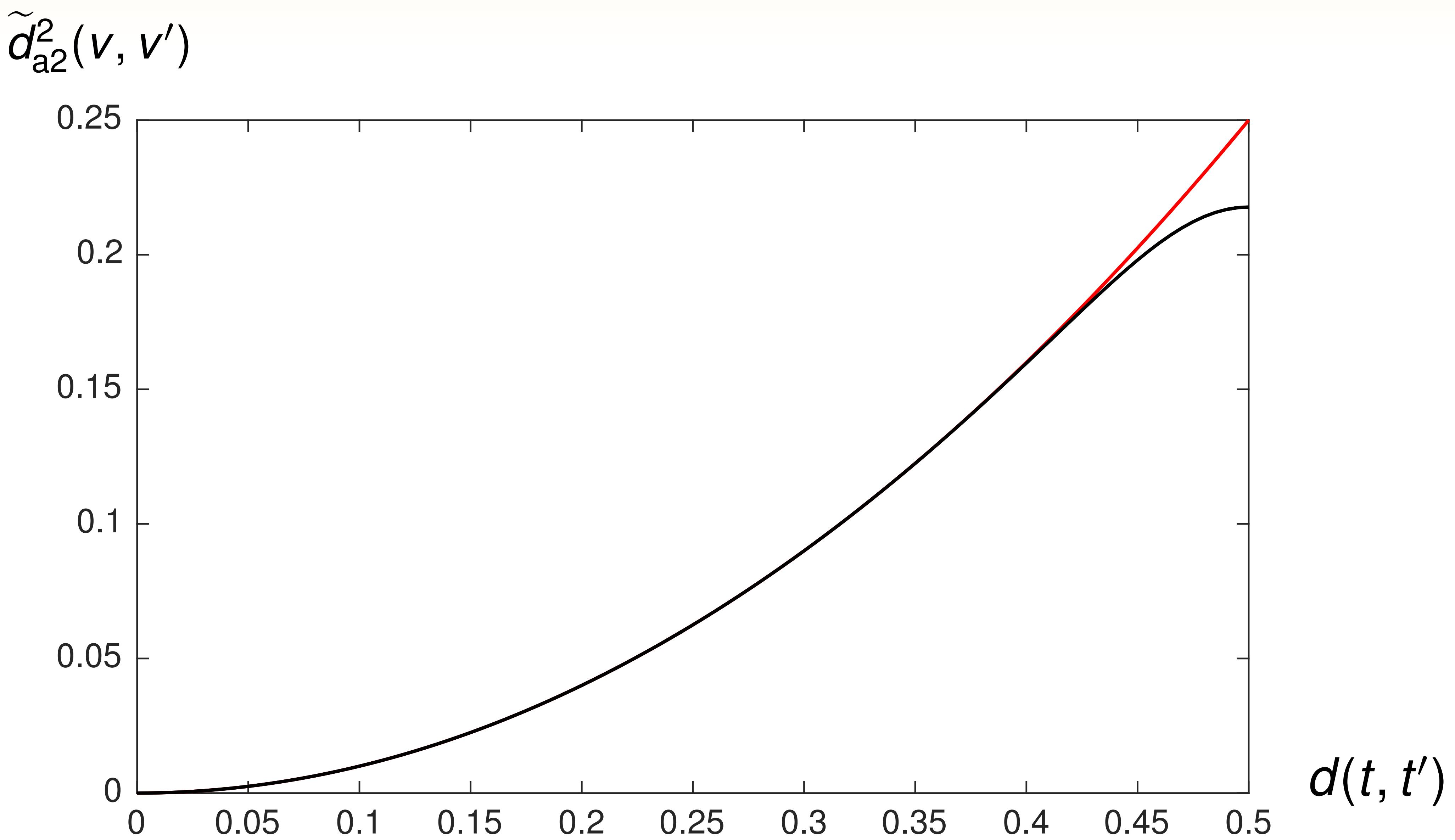
Explicit form :

set $w = \big((v_m - 2v'_m + {v'}_m^2 v_m^*)/(-4\pi^2 m^2)\big)_{m=-M}^{M}$,

with $w_0 = 0$ and $W = \mathrm{T}(w)$.

Then $\widetilde{d}_{a2}^2(a, v) = \mathrm{i}^+(-W)$

Atomic squared Wasserstein-2 distance



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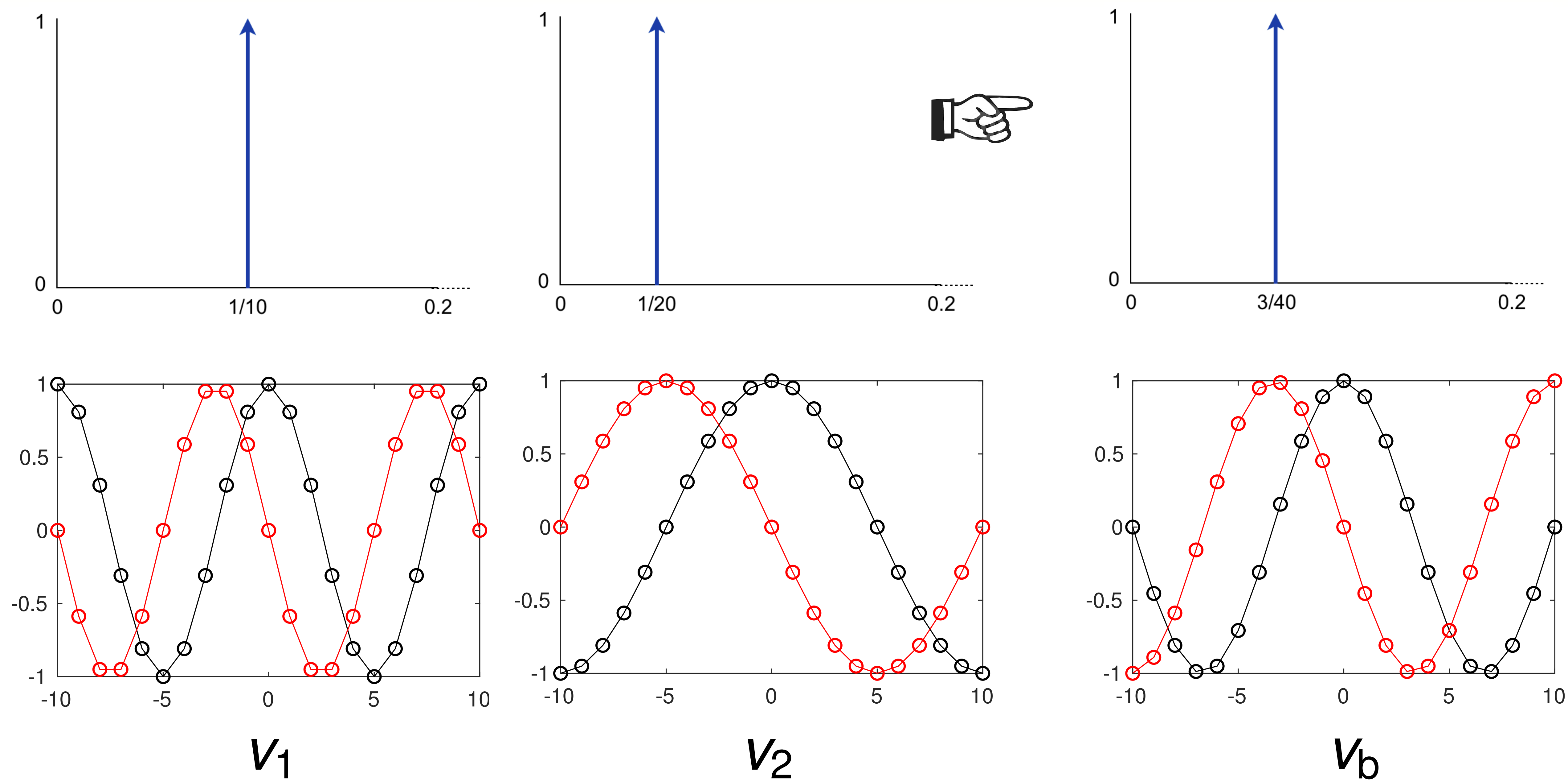
Wasserstein-2 barycenters



$$v_b = \operatorname*{arg\,min}_{v\,:\,T(v)\succcurlyeq 0} \widetilde{d}_{a2}^{2}(v, v_1) + \widetilde{d}_{a2}^{2}(v, v_2)$$

Application: Potts model

Piecewise-constant approximation with interface length regularization

$M = 8$

Thank you!