MAP recovery of polynomial splines from compressive samples and its application to vehicular signals

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ABSTRACT

We propose a stable reconstruction method for polynomial splines from compressive samples based on the maximum a posteriori (MAP) estimation. The polynomial splines are one of the most powerful tools for modeling signals in real applications. Since such signals are not band-limited, the classical sampling theorem cannot be applied to them. However, splines can be regarded as signals with finite rate of innovation and therefore be perfectly reconstructed from noiseless samples acquired at, approximately, the rate of innovation. In noisy case, the conventional approach exploits Cadzow denoising. Our approach based on the MAP estimation reconstructs the signals more stably than not only the conventional approach but also a maximum likelihood estimation. We show the effectiveness of the proposed method by applying it to compressive sampling of vehicular signals.

Keywords: Sparsity, signals with finite rate of innovation, B-spline, Cadzow denoising, particle swarm optimization (PSO), MAP estimation, vehicular signals

1. INTRODUCTION

Polynomial splines of degree n are piecewise polynomials satisfying a smooth constraint that imposes the continuity up to order $(n-1)$ at jointing points (knots).[1] Because of their ease of use, polynomial splines are used as a standard tool in signal and image processing, especially for interpolation. One difficulty of splines might arise from the fact that they are not band-limited. Because of this, the classical sampling theorem[2] cannot be applied to these signals. If splines are uniform, namely the jointing points are equally spaced, a generalized sampling theorem provides a way for perfect reconstruction under a proper choice of sampling functions.[3,4] If this is not the case, such a linear shift-invariant subspace approach does not apply any more.

A breakthrough was brought by the so-called sampling theory for signals with finite rate of innovation.[5–7] Typical examples of applications are radar, echo, or sonar. In these techniques, a radio or ultrasonic wave is transmitted to a target object, and the reflected waves enable us to compute the distance to the target. Since the transmitted pulse is known, such sparse signals can be expressed only using a few parameters, i.e., time delays and attenuation coefficients. Rate of innovation is defined by the degree of freedom of parameters per unit length. If it is finite, the signal is sampled at frequency near rate of innovation and is completely recovered from noiseless measurements. Since polynomial splines are also signals with finite rate of innovation, the same framework can be applied for sampling and recovery.

The standard framework can be summarized briefly as follows. The signal $s(t)$ is sampled using an appropriate kernel, like the Dirichlet kernel[5] or a Sum-of-Sincs.[8] Then, $s(t)$ can be perfectly reconstructed from the noiseless measurements using the annihilating filter technique (a.k.a. Prony's method).[5] If the measurements are corrupted by noise, the annihilating filter approach is used to produce a least squares solution.[9] To improve the performance of least squares, Cadzow denoising[10] is usually exploited.[9] This method tries to find a Toeplitz matrix of rank K by alternative projection. Since the set of matrices of rank K is not convex, however, any optimality or convergence are not guaranteed.[11] To solve this problem, the first and third authors proposed maximum likelihood approach, which works effectively in the regime of high

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signal-to-noise ratios (SNRs) in the sense that the Cramèr–Rao bound is mostly attained. On the other hand, for the case of low SNRs, maximum likelihood estimation can get unstable, sometimes produces huge coefficients.

In this paper, we proposed a more stable reconstruction method exploiting a maximum a posteriori (MAP) estimation, which penalizes huge coefficients by a quadratic regularization term. We keep the ideal performance of the maximum likelihood estimation for the high SNR regime by setting the regularization constant a small number, say 10^{-6} or 10^{-8}. The main difficulty in the MAP approach is that the criterion is not convex, like for the maximum likelihood estimation. It is possible to apply a convex optimization algorithm as a heuristic as in Ref 11. In this paper, however, we adopt a stochastic approach called *particle swarm optimization* (PSO),[12] because it is considered to be robust for getting trapped in local minima. Even though we have to pay some computational cost,[13,14] simulation results show that the proposed approach outperforms not only the least squares with Cadzow denoising, but also the maximum likelihood estimation using the same PSO.

This paper is organized as follows. Section 2 formulates the sampling setup that uses the Sum-of-Sincs kernel. Section 3 defines nonuniform polynomial splines as signals with finite rate of innovation. Section 4 proposes the MAP approach for recovery of the polynomial splines and show its effectiveness by simulations. Section 5 shows its application to vehicular signal compressive sampling. Section 6 concludes the paper.

2. SAMPLING FORMULATION

We first formulate the sampling setup in this paper, which discusses periodic signals $s(t)$, defined by

$$s(t) = \sum_{k' \in \mathbb{Z}} s_0(t - k'\tau), \tag{1}$$

where $s_0(t)$ is the signal in the interval $[0, \tau)$. This signal is sampled using a kernel $\psi(t)$ and yields N noiseless measurements

$$d_n = \langle s, \psi_n \rangle = \int_{-\infty}^{\infty} s(t)\overline{\psi(t - nT)}dt, \tag{2}$$

for $n = 0, \ldots, N-1$, $T = \tau/N$, and $\bar{z}$ stands for the conjugate of a complex number z. We assume that the measurements are corrupted by noise, as $y_n = d_n + e_n$, where $e_n \sim \mathcal{N}(0, \sigma^2)$ are independent random realizations of Gaussian noise.

We adopt for $\psi(t)$ the *Sum of Sincs* kernel,[8] which is defined in the frequency domain by

$$\hat{\psi}(\omega) = \frac{\tau}{\sqrt{2\pi}} \sum_{p=-P}^{P} b_p \operatorname{sinc}\left(\frac{\omega}{\frac{2\pi}{\tau}} - p\right), \tag{3}$$

where

$$\operatorname{sinc}(\omega) = \begin{cases} \sin(\pi\omega)/(\pi\omega) & (\omega \neq 0), \\ 1 & (\omega = 0). \end{cases}$$

Its time domain expression is

$$\psi(t) = \frac{\operatorname{rect}(t/\tau)}{\tau} \sum_{p=-P}^{P} b_p e^{i2p\pi t/\tau}, \tag{4}$$

where $\operatorname{rect}(t) = 1$ if $|t| \leq 0.5$ else 0 and $P \leq (N-1)/2$ is an integer. By setting $b_p = 1$ for all p, this kernel reduces to the standard Dirichlet kernel. Let $\hat{d}_p = \frac{1}{\tau}\int_0^\tau s(t)e^{-i2p\pi t/\tau}dt$ be the Fourier coefficients of $s(t)$. Then, it follows from Eq. (4) that

$$d_n = \sum_{p=-P}^{P} b_p \hat{d}_p e^{i2pn\pi/N}.$$

This admits the matrix representation

$$\boldsymbol{d} = F^{-1}B\hat{\boldsymbol{d}}, \tag{5}$$

where B is the diagonal matrix $\operatorname{diag}(b_{-P}, \ldots, b_P)$ and F is the discrete Fourier transform (DFT) matrix, defined accordingly. In a nutshell, the Fourier coefficients are related to the noiseless measurements acquired using the Sum of Sincs kernel *exactly* by the DFT.

3. NONUNIFORM POLYNOMIAL SPLINES AS FRI SIGNALS

Polynomial splines are special case of piecewise polynomials, which are defined as follows. For every $k = 0, \ldots, K-2$, let us define the function $\varphi_k(t)$ as

$$\varphi_k(t) = \begin{cases} v_k(t) & (t_k < t < t_{k+1}), \\ 0 & (\text{otherwise}), \end{cases}$$

and the function $\varphi_{K-1}(t)$ as

$$\varphi_{K-1}(t) = \begin{cases} v_{K-1}(t+\tau) & (0 \leq t < t_0), \\ v_{K-1}(t) & (t_{K-1} < t < \tau), \\ 0 & (\text{otherwise}), \end{cases}$$

where $v_k(t) = \sum_{r=0}^{R} \alpha_{k,r} t^r$. Then, a τ-periodic piecewise polynomial $s(t)$ of degree R is defined by $s(t)$ in (1) with

$$s_0(t) = \sum_{k=0}^{K-1} \varphi_k(t).$$

The piecewise polynomials are signals with finite rate of innovation, because $s(t)$ has K degrees of freedom from the positions t_k and RK from the coefficients $\alpha_{k,r}$ per period. This implies that the rate of innovation is $\rho = K(R+1)/\tau$.

The measurements of piecewise polynomials obtained by the Sum-of-Sincs kernel can be expressed by the parameters t_k and $\alpha_{k,r}$ as follows. Let us introduce matrices D, V_t, and $\tilde{V}_t$ as

$$D = \begin{pmatrix} & \mathbf{0} \\ \left(\frac{\tau}{i2\pi}\mathrm{diag}\left(\frac{1}{-P}, \frac{1}{-P+1}, \ldots, \frac{1}{P}\right)\right)^{R+1} & 1 \\ & \mathbf{0} \end{pmatrix}, \quad V_t = \begin{pmatrix} u_0^{-P} & \cdots & (-P)^R u_{K-1}^{-P} \\ u_0^{-P+1} & \cdots & (-P+1)^R u_{K-1}^{-P+1} \\ \vdots & \ddots & \vdots \\ u_0^{P} & \cdots & (P)^R u_{K-1}^{P} \end{pmatrix}, \quad \tilde{V}_t = \begin{pmatrix} V_t & \mathbf{0} \\ \mathbf{0}^T & 1 \end{pmatrix},$$

with $u_k = e^{-i2\pi t_k/\tau}$ and $\mathbf{0}$ indicating the zero vector. Note that the $R+1$th derivative of the piecewise polynomial in the sense of distribution is a sequence of derivatives of Diracs:[5]

$$s_0(t) = \sum_{k=0}^{K-1}\sum_{r=0}^{R-1} c_{k,r}\delta^{(r)}(t - t_k),$$

where $\delta^{(r)}(t)$ is the r th derivative of Dirac distribution. In this relation, the matrix D is the mapping from Fourier coefficients of the sequence of derivatives of Diracs to those of the piecewise polynomial. Further, we introduce a matrix W_t which maps $\alpha_{k,r}$ to the coefficients $c_{k,r}$ of the sequence of derivatives of Diracs. For instance when $R = 1$ and $K = 2$, W_t is given as

$$W_t = \begin{pmatrix} 0 & 0 & 1 & -1 \\ 1 & -1 & t_0 & -(t_0+\tau) \\ 0 & 0 & -1 & 1 \\ -1 & 1 & -t_1 & t_1 \\ \frac{t_1 - t_0}{\tau} & \frac{t_0+\tau-t_1}{\tau} & \frac{t_1^2 - t_0^2}{2\tau} & \frac{(t_0+\tau)^2 - t_1^2}{2\tau} \end{pmatrix}.$$

We refer to Ref. 15 for further details on the matrix W_t. Then, it holds that[15]

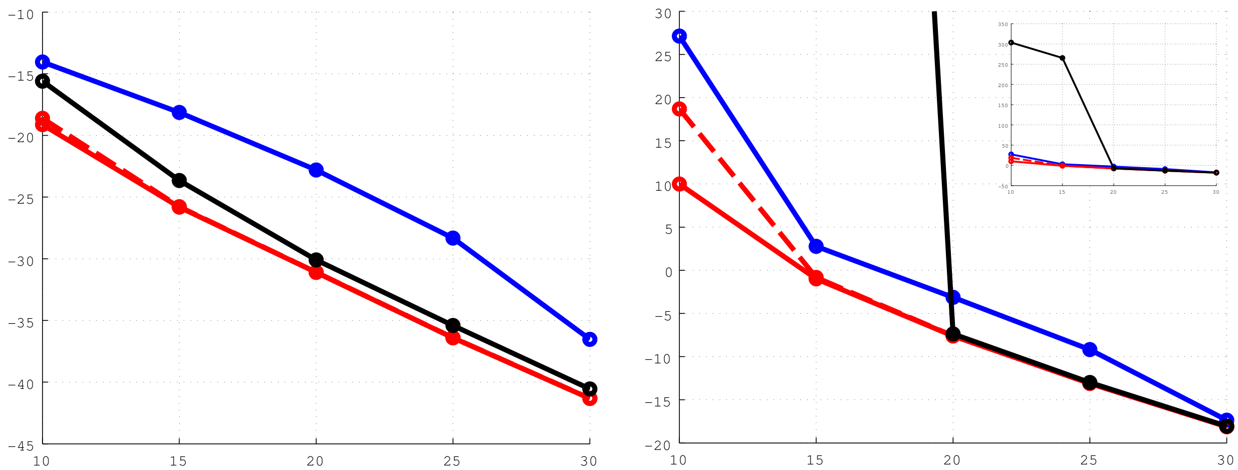
$$\boldsymbol{d} = F^{-1} B D \tilde{V}_t W_t \boldsymbol{\alpha},$$

where $\boldsymbol{\alpha} = (\alpha_{0,0} \ \cdots \ \alpha_{K-1,R})^{\mathrm{T}}$. That is, the noiseless measurements of the piecewise polynomial is expressed by using the locations t_k and the coefficients $\alpha_{k,r}$.

A polynomial spline is a piecewise polynomial, in which the coefficient $c_{k,r}$ with $0 < r < R$ of the corresponding sequence of derivatives of Diracs are zero. Let us denote a partial matrix of W_t by W_p, which consists of the rows corresponding to the coefficients $c_{k,r} = 0$. Then, the constraint can be expressed as $W_p\boldsymbol{\alpha} = 0$. This is satisfied any vector of the form of $\boldsymbol{\alpha} = (I - W_p^{\dagger} W_p)\bar{\boldsymbol{\alpha}}$, where I is the identity matrix, $W_p^{\dagger}$ is the Moore-Penrose generalized inverse,[16] and $\bar{\boldsymbol{\alpha}}$ is an arbitrary vector of the same dimension as $\boldsymbol{\alpha}$. In a nutshell, by using the auxiliary vector $\bar{\boldsymbol{\alpha}}$, measurements d_n are related from the parameters t_k and $\alpha_{k,r}$ as

$$\boldsymbol{d} = F^{-1} B \Phi_t \bar{\boldsymbol{\alpha}}, \tag{6}$$

where $\Phi_t = D\tilde{V}_t W_t (I - W_p^{\dagger} W_p)$.



(a) mean square errors for positions $\boldsymbol{t}$ (b) mean square errors for coefficients $\boldsymbol{\alpha}$

Figure 1. Mean square errors (MSEs) of (a) the positions $\boldsymbol{t} = (t_0, t_1)$ and (b) the coefficients $\boldsymbol{\alpha} = (\alpha_{0,0}, \alpha_{0,1}, \alpha_{1,0}, \alpha_{1,1})$, with respect to the SNR in dB. The red, blue and black lines show the results by the proposed method (MAP), least squares (LS) with Cadzow denoising, and maximum likelihood estimation, respectively.

4. MAP RECOVERY OF POLYNOMIAL SPLINES

As mentioned above, the $R + 1$th derivative of a polynomial spline is a sequence of Diracs. Hence, the classical approach first estimates this sequence and then reconstructs the polynomial spline by integration. This approach works exactly when the measurements are noiseless. When the measurements are noisy, Cadzow denoising is exploited. This method, however, does not guarantee any optimality. Thus, we can further improve noise resilience. Hence, we propose a direct estimation of the polynomial spline, without recasting the problem as the estimation of a sequence of Diracs. To this end, we exploit the maximum a posteriori (MAP) estimation as follows.

Because of (6), the log-likelihood function is defined as

$$L(\boldsymbol{t}, \bar{\boldsymbol{\alpha}}) = -\frac{\|\boldsymbol{y} - F^{-1} B \Phi_t \bar{\boldsymbol{\alpha}}\|^2}{2\sigma^2} - N \log(\sqrt{2\pi}\sigma).$$

Its maximization is equivalent to the minimization of the squared norm of $\|\hat{\boldsymbol{y}} - B\Phi_t \bar{\boldsymbol{\alpha}}\|^2$, where $\hat{\boldsymbol{y}} = F\boldsymbol{y}$. Through simulations, however, we found that minimization of only the squared norm provides huge value of $\boldsymbol{\alpha} = (I - W_p^\dagger W_p)\bar{\boldsymbol{\alpha}}$ occasionally for low signal-to-noise ratio (SNR). To prevent this phenomenon, we penalize huge coefficients by adding a quadratic regularization term to the squared norm, as

$$f_o(\boldsymbol{t}, \bar{\boldsymbol{\alpha}}) = \|\hat{\boldsymbol{y}} - B\Phi_t \bar{\boldsymbol{\alpha}}\|^2 + \lambda \|(I - W_p^\dagger W_p)\bar{\boldsymbol{\alpha}}\|^2. \tag{7}$$

Eq. (7) is quadratic with respect to $\bar{\boldsymbol{\alpha}}$, when $\boldsymbol{t}$ is fixed. Therefore, the optimal $\bar{\boldsymbol{\alpha}}$ for a fixed $\boldsymbol{t}$ is obtained analytically as

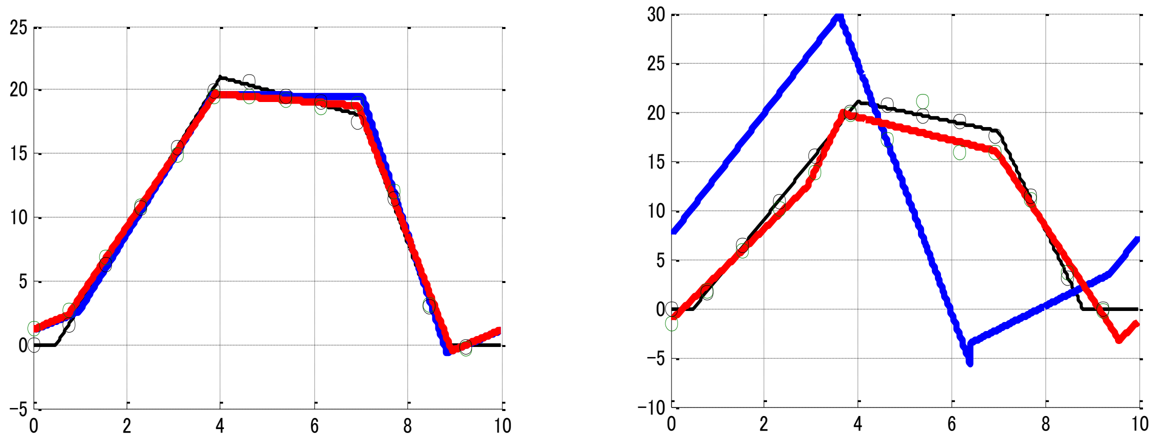
$$\bar{\boldsymbol{\alpha}}_t = \{\Phi_t^* B^* B \Phi_t + \lambda(I - W_p^\dagger W_p)\}^{-1} \Phi_t^* B^* \hat{\boldsymbol{y}}. \tag{8}$$

Hence, the minimizer of $f_o(\boldsymbol{t}, \bar{\boldsymbol{\alpha}})$ is found by searching $\boldsymbol{t}$ that minimizes

$$f(\boldsymbol{t}) = f_o(\boldsymbol{t}, \bar{\boldsymbol{\alpha}}_t) = \|\hat{\boldsymbol{y}} - B\Phi_t \bar{\boldsymbol{\alpha}}_t\|^2 + \lambda \|(I - W_p^\dagger W_p)\bar{\boldsymbol{\alpha}}_t\|^2$$

and then by computing $\bar{\boldsymbol{\alpha}}_t$ by (8). The coefficients $\boldsymbol{\alpha}$ are given by $(I - W_p^\dagger W_p)\bar{\boldsymbol{\alpha}}_t$.

The criterion $f(\boldsymbol{t})$ is non-convex and it is very difficult to find the global minimum solution. We thus exploit the so-called particle swarm optimization (PSO) algorithm.[12] The particles model the parameter $\boldsymbol{t}$ to be optimized. For each particle $j = 1, ..., J$, we first initialize the position $\boldsymbol{t}_j$ and its velocity $\dot{\boldsymbol{t}}_j$ with uniformly distributed random vectors in the domain. We use the particle's and swarm's best known positions $\boldsymbol{b}_j^{(p)}$ and $\boldsymbol{b}^{(s)}$, which are initialized by $\boldsymbol{t}_j$ and the best among the initial positions, respectively. Until a termination criterion is met, the particle's velocity $\dot{\boldsymbol{t}}_j$ and position $\boldsymbol{t}_j$ are updated by



(a) SNR = 25dB

(b) SNR = 15dB

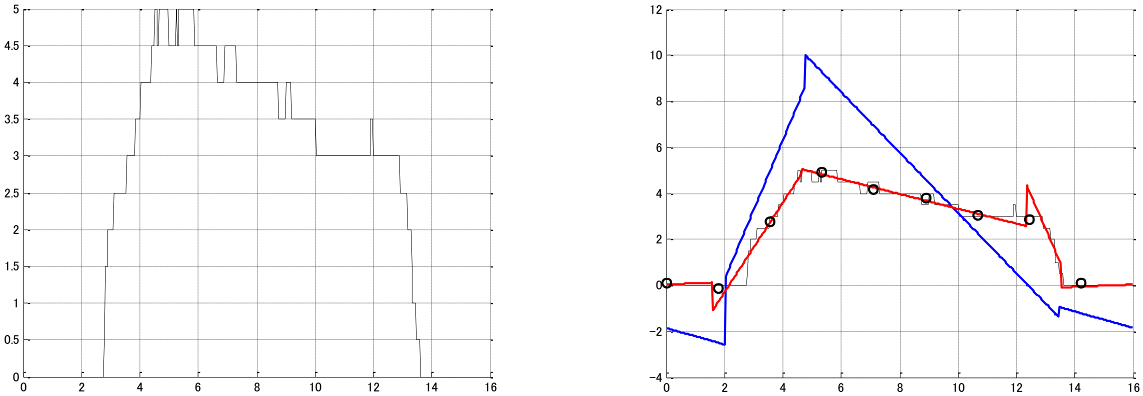
Figure 2. Mean square errors (MSE) of (a) the positions t_0, t_1 and (b) the coefficients $\alpha_{0,0}$, $\alpha_{0,1}$, $\alpha_{1,0}$, $\alpha_{1,1}$. The red, blue and black lines show the results by the proposed method (MAP), least squares (LS) with Cadzow denoising, and maximum likelihood estimation, respectively.

$$\dot{\boldsymbol{t}}_j \leftarrow w\dot{\boldsymbol{t}}_j + c_1 r_1(\boldsymbol{b}_j^{(p)} - \boldsymbol{t}_j) + c_2 r_2(\boldsymbol{b}^{(s)} - \boldsymbol{t}_j),$$
$$\boldsymbol{t}_j \leftarrow \boldsymbol{t}_j + \dot{\boldsymbol{t}}_j,$$

respectively, where c_1, c_2 are pre-defined constants near 1 and r_1, r_2 are uniform random variables within 0 and 1. If $f(\boldsymbol{t}_j) < f(\boldsymbol{b}_j^{(p)})$, then $\boldsymbol{b}_j^{(p)}$ is updated by $\boldsymbol{t}_j$. If $f(\boldsymbol{b}_j^{(p)}) < f(\boldsymbol{b}^{(s)})$, then $\boldsymbol{b}^{(s)}$ is replaced by $\boldsymbol{b}_j^{(p)}$. Finally, $\boldsymbol{b}^{(s)}$ gives the best found solution. Because of its global and random nature, PSO is more robust than gradient approaches, against getting trapped in local minima. Meanwhile, its drawback might be a relatively high computational cost caused by its random nature.

The performance of the proposed method was evaluated by simulations. The target signal is a $\tau = 1$-periodic piecewise polynomial of degree $R = 1$ with $K = 2$ knots. The unknown parameters are $\boldsymbol{t} = (0.2, 0.6)$ and $\boldsymbol{\alpha} = (\alpha_{0,0}, \alpha_{0,1}, \alpha_{1,0}, \alpha_{1,1}) = (-1, 3, 2, -2)$. For PSO, we used $J = 150$ particles and $(w, c_1, c_2) = (0.4, 0, 9, 0.4)$, $(0.9, 0.4, 0.4)$ and $(0.4, 0.4, 0.9)$ for 75, 45 and 30 particles, respectively. One thousand noise vectors $\boldsymbol{e}$ were generated from the Gaussian distribution in which σ was determined so that the SNR, defined by $10\log_{10}\frac{\|\boldsymbol{d}\|^2}{\sigma^2 N}$, becomes 10, 15, . . ., 30[dB]. For each experiment, we computed estimates $\hat{\boldsymbol{t}}$ and $\hat{\boldsymbol{c}}$ of $\boldsymbol{t}$ and $\boldsymbol{c}$, for 1,000 different noise realizations. Accordingly, the mean square errors for $\boldsymbol{t}$ and $\boldsymbol{\alpha}$ were defined as the average over the 1,000 trials of $\|\hat{\boldsymbol{t}} - \boldsymbol{t}\|^2$ and $\|\hat{\boldsymbol{c}} - \boldsymbol{c}\|^2$, respectively. The results are shown in Fig. 1, where the red, blue and black lines show the results by the proposed method (MAP), least squares (LS) with Cadzow denoising,[9] and maximum likelihood estimation with PSO, respectively. The solid and dashed red lines show the results by the proposed method with $\lambda = 10^{-6}$ and 10^{-8}, respectively. We can see that the proposed method outperforms the conventional methods for every value of the SNR. Note that the maximum likelihood estimation works as well as the proposed approach for high SNR values. For SNRs lower than 20dB, the performance suddenly degraded for coefficients. The small box in the top right corner shows the entire shape of each curve. The main reason for the degradation is that in our approach, maximum likelihood estimation requires the generalized inverse of $B\Phi_t$, which gets unstable when the estimates for t_0 and t_1 are very close. This effect is sufficiently suppressed by the regularization by the quadratic term. On the other hand, if we search for $\boldsymbol{t}$ and $\bar{\alpha}$ independently, then the performance of the maximum likelihood estimation might get better than the results in the figure. One drawback of this approach is that we have to search more parameters than the current approach. The regularization parameter for the proposed method is determined only by trials. Its elaborate tuning will improve the performance.

An example of recovery with $K = 4$, $R = 1$, and $N = 13$ is shown in Fig. 2. The black curve shows the target Bspline of degree 1. The black circles denote noiseless measurements. The noisy measurements are indicated by green circles. The SNR is 25dB in Figure (a) and 15dB in (b). The red and black curves are the results by the proposed method with $\lambda = 10^{-6}$ and LS with Cadzow denoising, respectively. We can see that the proposed method gives better results than



(a) Original data (b) reconstructed signals from compressive measurements

Figure 3. Application to vehicular signal compressive sensing. (a) signal of an accelerator pedal (gas pedal) open degree in percent, acquired during a drive with a sampling interval of 0.032 second. The number of samples is 500. (b) reconstructed signal from $N = 9$ compressive samples downsampled by Direchlet kernel (98.2% compression rate). The red and blue lines show the reconstructed signals by the proposed method and the LS with Cadzow denoising, respectively. The squared errors are 6.8% for the proposed method and 53.1% for the conventional method.

the conventional approach. Note that when SNR=25dB, the LS with Cadzow denoising worked as well as the proposed method. When SNR=15dB, however, the LS with Cadzow denoising got unstable, while the proposed method provided stable recovery.

5. APPLICATION TO VEHICULAR SIGNAL COMPRESSIVE SENSING

We applied our approach to compressive sampling of vehicular signals. The fundamental technology in the intelligent transport system (ITS) is data transmission between vehicles, between vehicles and servers, or between vehicles and pedestrians. Increase of transmission channels is a threat to communications infrastructure. Hence, we wish to reduce the data amount with keeping its quality. Figure 3 (a) shows a signal of an accelerator pedal (gas pedal) open degree in percent. This signal was actually acquired during a drive with a sampling interval of 0.032 second. The number of samples is 500. The signal is quantized, hence looks like a step signal.

To produce compressive measurements, we computed (2) approximately by sum of the product between the original samples in (a) and Dirichlet kernel (the kernel in (3) with $b_p = 1$). The circles in Figure (b) shows those measurements. Now, the number of measurements was reduced down to $N = 9$. The compression rate is 98.2%. From these measurements, we reconstructed the target signal by using the proposed method with $\lambda = 10^{-6}$ as well as the LS with Cadzow denoising. The former and latter are shown by the red and blue lines, respectively. We can see that the proposed method provides much better result than the conventional approach. The squared error between the original signal and the reconstructed signal were 6.8% for the proposed method and 53.1% for the conventional method. These values also show the effectiveness of the proposed method. In this simulations, the parameter K was determined by the programmer, while in real scenarios K has to be determined automatically. We have to device a method to do it in near future.

6. CONCLUSION

We proposed a stable reconstruction method for polynomial spline signals from compressive samples based on the maximum a posteriori (MAP) estimation, which is equivalent to the minimization of the sum of a data fidelity term and a quadrature regularization term. Since the criterion is not convex, it is very difficult to find the optimum solution. To this end, we exploited a heuristic stochastic approach called PSO. The simulation results showed that the proposed method outperformed not only the conventional approach based on least squares with Cadzow denoising but also a maximum likelihood estimation with PSO. We applied the proposed method to the acceleration pedal signal, which has discontinuous change of values. We compressed the acquired data 98.2% and recovered the original data with an error less than 7%. To device a method to determine the parameter K automatically is one of the our future tasks.

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