FAST RECONSTRUCTION FROM NON-UNIFORM SAMPLES IN SHIFT-INVARIANT SPACES

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ABSTRACT

We propose a new approach for signal reconstruction from non-uniform samples, without constraints on their locations. We look for a function that belongs to a linear shift-invariant space, and minimizes a variational criterion that is a weighted sum of a least-squares data term and a quadratic term penalizing the lack of smoothness. This leads to a resolution-dependent solution, that can be computed exactly by a fast non-iterative algorithm.

1. INTRODUCTION

The representation of sampled data by means of a continuous model is essential for common tasks such as interpolation and resampling. Traditional methods achieve perfect reconstruction of a signal from non-uniform samples, under strong contraints on the samples locations [1, 2, 3, 4, 5]. If these conditions are not met, one has to give up perfect reconstruction, and seek a smooth function that is a satisfying model for the observed samples.

In this article, we propose a novel approach for unidimensional signal reconstruction from non-uniform samples, without any constraint on their locations. We adopt a variational approach where the reconstruction is formulated as the minimization of a cost depending on two terms: a least-squares data term on one side, and a smoothness quadratic functional on the other side. The reconstructed function is contrained to lie in a linear shift-invariant space, which ensures that the solution is parameterized by coefficients attached to a uniform reconstruction grid with step T, and not to the data locations. Thus, we aim at reconstructing a continuous-time function $f_T(t)$ that depends on a resolution parameter T. Non-uniform to uniform resampling is a straightforward application of our approach, where T is simply matched to the resolution of the uniform target lattice.

Besides its theoretical advantages, the proposed method is computationally attractive. We present a fast, non-iterative algorithm, that computes the coefficients determining the reconstructed function. This original algorithm performs a two-pass time-varying recursive filtering on the data.

The paper is organized as follows. We define the problem and derive its solution in Sect. 2. We then propose a practical algorithm for computing the solution in Sect. 3. Finally, in Sect. 4, we discuss the properties of the reconstructed function, and we present experimental results.

Throughout this paper, we use some notational conventions: parentheses are used for continuous-time signals, *e.g.*, $f(t)$, and brackets for discrete time signals, *e.g.*, $s = (s[n])_{n\in\mathbb{Z}}$. We define the $z-$transform of a discrete signal s as $S(z) = \sum_{n\in\mathbb{Z}} s[n]z^{-n}$. Continuous and discrete convolutions are denoted by $*$.

2. VARIATIONAL RECONSTRUCTION IN A LINEAR SHIFT-INVARIANT SPACE

2.1 Problem statement

Let us assume we have a finite number N of measurements $(s[n])_{n\in[0,N-1]}$ at locations $(x[n])_{n\in[0,N-1]}$ within a finite interval $\mathcal{I}$. Typically, the data are non-uniform samples of an unknown function $s(t)$: $s[n] = s(x[n])$ for every n. We want to reconstruct a continuous-time function $f(t)$, defined for every $t \in \mathcal{I}$, that modelizes the discrete data.

The classical variational approach consists in solving a minimization problem:

$$f = \underset{g\in H^r}{\operatorname{argmin}} \left(\sum_{n=0}^{N-1} \left|g(x[n]) - s[n]\right|^2 + \lambda \int_{\mathcal{I}} |g^{(r)}(t)|^2\, dt \right). \quad (1)$$

where $g^{(r)}$ is the r^{th} derivative of g, for some integer $r \geq 1$, and H^r is the Sobolev space of order r [6]. This criterion is composed of two antagonist terms, one controlling the closeness to the data, the other one enforcing the solution to be smooth. The parameter $\lambda > 0$ is a Lagrangian parameter working as a tradeoff factor between these two terms. The integer r controls the smoothness of the reconstruction: the values $r = 1$ and $r = 2$ are the most frequently used, and correspond to searching a function that has maximum flatness, and minimum curvature, respectively.

The solution to this variational problem can be expressed as $f(t) = \sum_{n=0}^{N-1} c[n]\, |t - x[n]|^{2r-1} + p(t)$ [7]. It is made of a polynomial $p(t)$ of degree less than r and a linear combination of *radial basis functions* (RBF) $|t|^{2r-1}$ positioned at the sampling locations $x[n]$. This implies that the solution $f(t)$ is a non-uniform polynomial spline of degree $2r-1$ with knots at the $x[n]$.

2.2 Reconstruction in a shift-invariant space

In this work, we propose to minimize the same criterion as in (1), but in a *linear shift-invariant* (LSI) space: if we let $T > 0$ be an arbitrary real number, the LSI space $V_T(\varphi)$ is spanned by the shifts of a generating function $\varphi(\frac{t}{T})$:

$$V_T(\varphi) = \Big\{ \sum_{k\in\mathbb{Z}} c_T[k]\, \varphi(\frac{t}{T} - k) \ : \ (c_T[k]) \in \mathbb{R}^{\mathbb{Z}} \Big\}. \quad (2)$$

So, we look for a function $f_T(t) \in V_T(\varphi)$ having the form

$$f_T(t) = \sum_{k\in\mathbb{Z}} c_T[k] \varphi(\frac{t}{T} - k), \quad (3)$$

where the discrete coefficients $c_T[k]$ are the unknowns to be determined, so as to minimize the criterion in (1). In other words, our approach consists in choosing the reconstruction space $V_T(\varphi)$ *a priori*, and to look for the better function in this space, in the sense of the criterion in (1).

For the problem to be well posed, we have to make some hypotheses. First, we assume that φ is bounded and has compact support (included in the interval $(-W,W)$ for some $W\in\mathbb{N}$), and that $\int_{\mathbb{R}}|\varphi^{(r)}(t)|^2dt<\infty$. We also assume that the functions $\{\varphi(\frac{t}{T}-k)\}$ form a Riesz basis of $V_T(\varphi)\cap L_2(\mathbb{R})$. Moreover, we suppose that there are at least r distinct locations $x[n]$.

Although φ may be almost arbitrary, we adopt the choice $\varphi=\beta^{2r-1}$, the centered B-spline of degree $2r-1$ [8]. Thus, f_T is a uniform polynomial spline of degree $2r-1$, with knots at the Tk. With this choice, when the samples are uniform at the locations $x[n]=Tn$, our solution coincides with the global solution in (1), that is, f_T minimizes the criterion not only in $V_T(\varphi)$, but also in the whole space H^r.

So, our method depends on three parameters r, T, λ, whose influence will be discussed in Sect. 4.

2.3 Solution to the minimization problem

Finding the reconstructed function $f_T(t)$ amounts to determining the sequence c_T in (3) so that the cost function in (1) is minimized. In order to express this cost $\Psi(c_T)$, we first rewrite the data fidelity term as a function of c_T:

$$\sum_{n=0}^{N-1}\left|f_T(x[n])-s[n]\right|^2=\sum_{n=0}^{N-1}\left(s[n]-\sum_{k\in\mathbb{Z}}c_T[k]\varphi(\frac{x[n]}{T}-k)\right)^2. \tag{4}$$

Now, we rewrite the variational term as a function of c_T. We look for a reconstruction in the finite interval $\mathcal{I}$, assumed for convenience to have the form $\mathcal{I}=[0,KT]$ for some $K\in\mathbb{N}$. Since f_T is completely determined within $\mathcal{I}$ by the coefficients $c_T[k]$, $k\in[-W+1,K+W-1]$, we set $c_T[k]=0$ for $k\notin[-W+1,K+W-1]$. First, let us write the integral over $\mathbb{R}$, and not $\mathcal{I}$. To this aim, we introduce the autocorrelation of φ: $a_\varphi(t)=\bar{\varphi}*\varphi(t)$, using the flip operator $\bar{\varphi}(t)=\varphi(-t)$. We then introduce the discrete sequence $q_{\varphi,r}$ defined by $q_{\varphi,r}[k]=\frac{(-1)^r}{T}a_\varphi^{(2r)}(k)$. Since $\varphi^{(r)}(-t)=(-1)^r\bar{\varphi}^{(r)}(t)$, and differentiations commute with convolutions, we have:

$$\begin{aligned}\int_{\mathbb{R}}\left|f_T^{(r)}(t)\right|^2dt&=\frac{1}{T^2}\int_{\mathbb{R}}\left(\sum_{k\in\mathbb{Z}}c_T[k]\varphi^{(r)}(\frac{t}{T}-k)\right)^2dt\\&=\frac{1}{T}\sum_{k,l\in\mathbb{Z}}c_T[k]c_T[l]\int_{\mathbb{R}}\varphi^{(r)}(x-(k-l))\varphi^{(r)}(x)\,dx\\&=\frac{(-1)^r}{T}\sum_{k,l\in\mathbb{Z}}c_T[k]c_T[l]\,(\bar{\varphi}^{(r)}*\varphi^{(r)})\,(k-l)\\&=\frac{(-1)^r}{T}\sum_{k,l\in\mathbb{Z}}c_T[k]c_T[l]\,a_\varphi^{(2r)}(k-l)\\&=\sum_{k\in\mathbb{Z}}c_T[k]\,(c_T*q_{\varphi,r})[k].\end{aligned} \tag{5}$$

Since we concentrate on spline reconstruction ($\varphi=\beta^{2r-1}$), we can give the general form of the corresponding filter $q_{\varphi,r}$. B-splines verify the simple relation $a_{\beta^{2r-1}}(t)=\beta^{4r-1}(t)$, and the derivative of a spline is also a spline of lower degree [8]. Using these properties, we get for $r=1$ and $r=2$, respectively, $Q_{\beta^1,1}(z)=(-z+2-z^{-1})/T$ and $Q_{\beta^3,2}(z)=(z^3-9z+16-9z^{-1}+z^{-3})/6T$.

It is convenient to express the cost $\Psi(c_T)$ in terms of matrices and vectors

$$\Psi(\mathbf{c})=\|\mathbf{Mc}-\mathbf{s}\|^2+\lambda\mathbf{c}^{\mathrm{T}}\mathbf{Qc}, \tag{6}$$

using the following quantities ($\cdot^{\mathrm{T}}$ is the transpose operator):

$$\begin{aligned}\mathbf{c}&=\left[c_T[-W+1]\ \cdots\ c_T[K+W-1]\right]^{\mathrm{T}},\\\mathbf{s}&=\left[s[0]\ s[1]\ \cdots\ s[N-1]\right]^{\mathrm{T}},\\\mathbf{M}&=[M[n,k]]\text{ with }M[n,k]=\varphi(\frac{x[n]}{T}-k),\\&\qquad\text{for }n\in[0,N-1],k\in[-W+1,K+W-1],\\\mathbf{Q}&=[Q[k,l]],\ \text{for }k,l\in[-W+1,K+W-1],\\&\qquad\text{with }Q[k,l]=q_{\varphi,r}[k-l]=\frac{(-1)^r}{T}a_\varphi^{(2r)}(k-l),\end{aligned} \tag{7}$$

except for the first and last rows of $\mathbf{Q}$ that contain particular values, because $\Psi(c_T)$ is defined with the integral over $\mathcal{I}$, and not $\mathbb{R}$ as in (5). These special values are in squares of size $(2W-1)^2$ in the lower-left and upper-right corners of the matrix. To compute them for the left boundary (this would be the same for the right one), we have to develop the left-hand side of the following equality, and identify the coefficients with its right-hand side:

$$\int_0^{2W-1}\Big|\sum_{k=-W+1}^{W-1}c_T[k]\varphi^{(r)}(\frac{t}{T}-k)\Big|^2dt=\sum_{k,l=-W+1}^{W-1}Q[k,l]c_T[k]c_T[l]. \tag{8}$$

For instance, in the cases $\varphi=\beta^1, r=1$ and $\varphi=\beta^3, r=2$, $\mathbf{Q}$ takes the respective forms:

$$\frac{1}{T}\begin{bmatrix}\mathbf{1}&-1&0&0&0&\cdots\\-1&2&-1&0&0&\ddots\\0&-1&2&-1&0&\ddots\\0&0&-1&2&-1&\ddots\\0&0&0&-1&2&\ddots\\\vdots&\ddots&\ddots&\ddots&\ddots&\ddots\end{bmatrix},\ \frac{1}{6T}\begin{bmatrix}\mathbf{2}&\mathbf{-3}&\mathbf{0}&1&0&\cdots\\\mathbf{-3}&\mathbf{8}&\mathbf{-6}&0&1&\ddots\\\mathbf{0}&\mathbf{-6}&\mathbf{14}&-9&0&\ddots\\1&0&-9&16&-9&\ddots\\0&1&0&-9&16&\ddots\\\vdots&\ddots&\ddots&\ddots&\ddots&\ddots\end{bmatrix}. \tag{9}$$

Now, let us define $\mathbf{A}=\mathbf{M}^{\mathrm{T}}\mathbf{M}+\lambda\mathbf{Q}$ and $\mathbf{y}=\mathbf{M}^{\mathrm{T}}\mathbf{s}$. Minimizing the cost $\Psi(\mathbf{c})$ amounts to solving the linear system

$$\mathbf{Ac}=\mathbf{y}. \tag{10}$$

or equivalently, the set of equations, for $k\in[-W+1,K+W-1]$:

$$\sum_{l\in\mathbb{Z}}\left[\sum_{n=0}^{N-1}\varphi(\frac{x[n]}{T}-k)\varphi(\frac{x[n]}{T}-l)+\lambda\,Q[k,l]\right]c_T[l]=\sum_{n=0}^{N-1}\varphi(\frac{x[n]}{T}-k)s[n]. \tag{11}$$

The linear system in (10) has a unique and well defined solution, since $\mathbf{A}$ is symmetric and positive definite. We now exploit its particular structure for solving it efficiently, without expliciting the underlying matrices.

3. EFFICIENT IMPLEMENTATION

3.1 Strategy

We have seen that our problem boils down to solving the linear system $\mathbf{Ac} = \mathbf{y}$ that is symmetric and positive definite. Since φ has compact support, this system is also band-diagonal with only $4W-1$ diagonals containing non-zero entries. Thus, an efficient resolution strategy is as follows:

1. We perform the Cholesky decomposition of $\mathbf{A}$ [9]; that is, we look for the lower triangular matrix $\mathbf{L}$ such that $\mathbf{A} = \mathbf{LL}^{\mathrm{T}}$. $\mathbf{L}$ is band-diagonal with $2W$ diagonals containing non-zero entries (*i.e.* $L[k,l] \neq 0$ only if $-2W < k-l \leq 0$).
2. We solve the lower triangular system $\mathbf{L}\mathring{\mathbf{c}} = \mathbf{y}$ by forward substitution [9].
3. We finally solve the upper triangular system $\mathbf{L}^{\mathrm{T}}\mathbf{c} = \mathring{\mathbf{c}}$ by backward substitution [9].

Let us detail the practical implementation of these three steps. The Cholesky decomposition is performed row by row by a fast algorithm that takes advantage of the band-diagonal structure of $\mathbf{A}$. In fact, the decomposition is straightforward once the equality $\mathbf{A} = \mathbf{LL}^{\mathrm{T}}$ is decomposed, which yields:

$$L[k,k] = \Big(A[k,k] - \sum_{j=-2W+1}^{-1} L[k,k+j]^2\Big)^{1/2}, \quad (12)$$

For $i = 1 \ldots 2W-1$, $L[k+i,k] =$

$$\frac{1}{L[k,k]}\Big(A[k,k+i] - \sum_{j=i-2W+1}^{-1} L[k,k+j]L[k+i,k+j]\Big). \quad (13)$$

If these two equations are evaluated in the increasing order $k = k_{\min} \ldots k_{\max}$, it appears that the entries of $\mathbf{L}$ that occur on the right-hand side are already determined at the time they are needed. Once these values have been calculated, the first triangular system is solved by: for $k = k_{\min} \ldots k_{\max}$,

$$\mathring{c}_T[k] = \frac{1}{L[k,k]}\Big(y[k] - \sum_{i=-2W+1}^{-1} L[k,k+i]\mathring{c}_T[k+i]\Big), \quad (14)$$

and the second linear system is solved by: for $k = k_{\max} \ldots k_{\min}$,

$$c_T[k] = \frac{1}{L[k,k]}\Big(\mathring{c}_T[k] - \sum_{i=1}^{2W-1} L[k+i,k]c_T[k+i]\Big). \quad (15)$$

Now, we have all the ingredients to give the practical algorithm that computes the coefficients $c_T[k], k \in [-W+1, K+W-1]$.

3.2 Practical algorithm

It is possible to perform the Cholesky decomposition (12), (13) and the forward substitution (14) in a single forward pass. If the samples are ordered such that $x[n+1] \geq x[n]$ for every n, the data set $(x[n], s[n])$ can be accessed progressively. Hence, this pass can be computed on-the-fly, as the data are made available. If the data locations are not sorted, or if $N >> K$, the computation time will be consumed mostly for accessing the data. In these cases, or if it is not useful to compute the coefficients online from the data, it is much more appropriate to use the following version of the algorithm, that consists in three passes. Let us define the auxiliary variables $a[i] = A[k,k+i]$ and $u[k,i] = L[k+i,k]$. The first pass is performed on the data, so as to construct the upper part of the matrix $\mathbf{A}$ (that contains all the information since $\mathbf{A}$ is symmetric), that is stored temporarily in the coefficients $u[k,i]$. The $y[k]$ are stored in the $\mathring{c}_T[k]$. During the second pass, the Cholesky decomposition and the forward substitution are performed in place, that is, the $u[k,i]$ and $\mathring{c}_T[k]$ take their true values. Finally, during the third pass, the backward substitution (15) is performed.

- First pass:
    - for k from $-W+1$ to $K+W-1$ {
        - $\mathring{c}_T[k] = 0$;
        - for i from 0 to $\min(2W-1, K+W-1-k)$,
            - $u[k,i] := \lambda\, Q[k,k+i]$;
    - }
    - for n from 0 to $N-1$,
        - for k from $\lfloor \frac{x[n]}{T} + 1 - W \rfloor$ to $\lceil \frac{x[n]}{T} - 1 + W \rceil$ {
            - for i from 0 to $\min(2W-1, K+W-1-k)$,
                - $u[k,i] := u[k,i] + \varphi(\frac{x[n]}{T} - k)\varphi(\frac{x[n]}{T} - k - i)$;
            - $\mathring{c}_T[k] := \mathring{c}_T[k] + \varphi(\frac{x[n]}{T} - k)s[n]$;
        - }
- Second pass:
    - for k from $-W+1$ to $K+W-1$ {
        - $i_{\min} := \max(-2W+1, -W+1-k)$;
        - $i_{\max} := \min(2W-1, K+W-1-k)$;
        - $u[k,0] := \Big(u[k,0] - \sum_{i=i_{\min}}^{-1} u[k+i,-i]^2\Big)^{1/2}$;
        - $\mathring{c}_T[k] := \frac{1}{u[k,0]}\Big(\mathring{c}_T[k] - \sum_{i=i_{\min}}^{-1} u[k+i,-i]\mathring{c}_T[k+i]\Big)$;
        - for i from 1 to $i_{\max}$,
            - $u[k,i] := \frac{1}{u[k,0]}\Big(u[k,i] - \sum_{j=\max(i-2W+1,-W+1-k)}^{-1} u[k+j,-j]\,u[k+j,i-j]\Big)$;
    - }
- Third pass:
    - for k from $K+W-1$ down to $-W+1$ {
        - $i_{\max} := \min(2W-1, K+W-1-k)$;
        - $c_T[k] := \frac{1}{u[k,0]}\Big(\mathring{c}_T[k] - \sum_{i=1}^{i_{\max}} u[k,i]c_T[k+i]\Big)$;
    - }

This algorithm can be interpreted as a two-pass time-varying recursive filtering, with filters computed on-the-fly by the Cholesky decomposition. This decomposition is very stable numerically [9]. However, the condition number of the linear system depends on the sampling locations. A round-off error that occurs on a coefficient $c_T[k]$ can propagate to its neighbors inside a large region without samples, but its amplitude, limited to the machine accuracy, will not grow. Therefore, this is not a problematic issue.

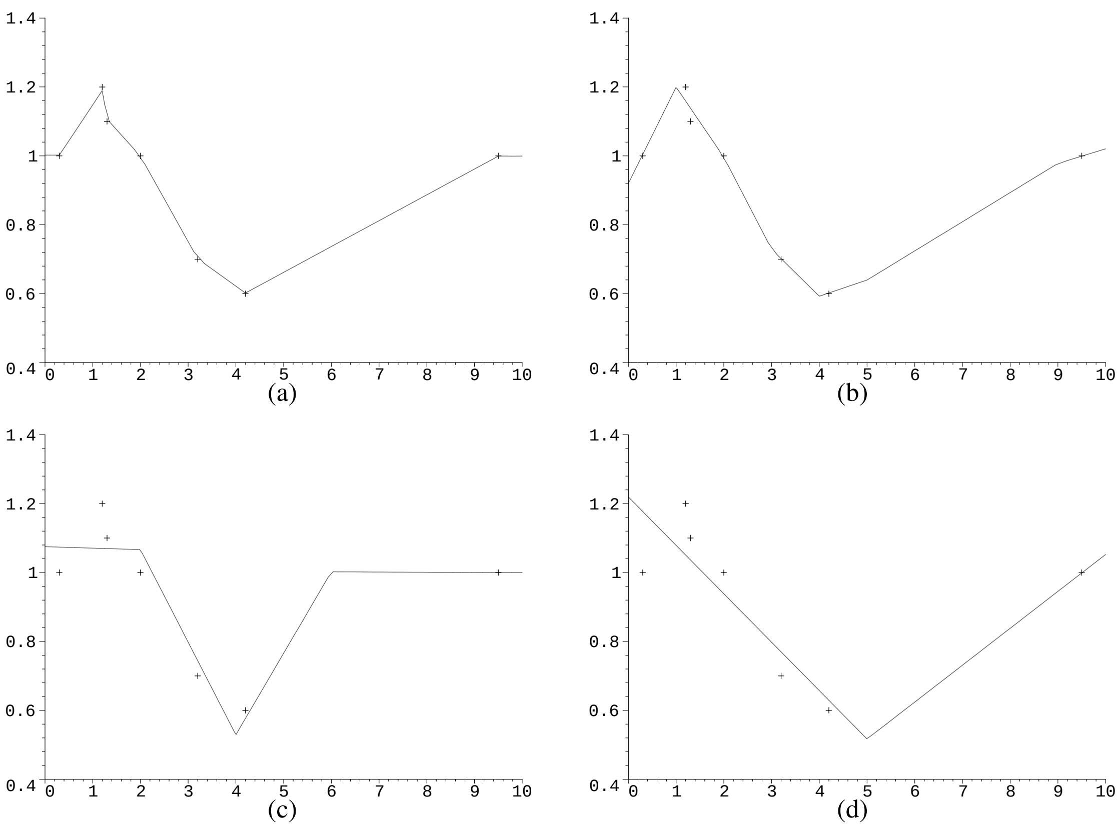

Figure 1: Uniform linear splines with different resolutions ($\varphi = \beta^1$, $r = 1$, $\lambda = 0.01$) fitted on 7 point samples in the interval $[0,10]$. (a): $T = 0.1$. (b): $T = 1$. (c): $T = 2$. (d): $T = 5$. Each spline has $10/T + 1$ knots at the Tk, $k \in [0, 10/T]$. When $T \to 0$, f_T approaches the non-uniform smoothing spline of degree 1, which has its knots at the non-uniform sampling locations.

3.3 Computation time and storage requirements

The coefficients $c_T[k]$ can be computed in place, replacing the intermediate values $\mathring{c}_T[k]$. Apart from the memory needed for the results $c_T[k]$, $2W(K+2W-1)$ units of auxiliary storage are required for the coefficients $u[k,i]$, that are calculated during the second pass and used in the third one.

The computation time of the proposed algorithm is made of: $O(W^2N)$, $O(W^2K)$, $O(WN)$ for the calculation of the elements in $\mathbf{A}$, $\mathbf{L}^{\mathrm{T}}$ and $\mathbf{y}$ respectively, and $O(KW)$ for the forward and backward substitutions (where $O(x)$ stands for "proportional to x"). So, the total time reduces to $O(W^2(N+K))$; it is linear in N and K, which is particularly attractive. An implementation in C language of the proposed implementation, running on a 1.6 GHz laptop PC, gives a computation time of 0.001s for a reconstruction from $N = 10000$ samples randomly located in the interval $[0,100]$ ($K = 100$, $T = 1$).

4. INFLUENCE OF THE PARAMETERS

The parameter r controls the kind of smoothness that is enforced on the solution. In the simpler case $r = 1, \varphi = \beta^1$ illustrated in Fig. 1, the reconstruction is piecewise linear, with knots at the Tk, $k \in \mathbb{Z}$, which means that $f_T(t)$ is linear on each interval $[Tk, T(k+1)]$. If $r = 2, \varphi = \beta^3$, as in Fig. 2, the reconstruction is smoother: it is twice continuously differentiable.

The parameter T controls the coarseness of the representation. When reconstructing a signal over an interval $[0,S]$, we obtain a parametric solution with $K+1 = S/T+1$ degrees of freedom. If this representation has to be sparse, *e.g.* in coding applications, or if the computation time is limited, then T will be choosen relatively large. Conversely, when $T \to 0$, the solution f_T becomes closer and closer to the non-uniform solution in the RBF framework. The influence of T is illustrated in Fig. 1, with $\varphi = \beta^1, r = 1$. In fact, there is a one-to-one correspondence between the coefficients $c_T[k]$ and the point values $f_T(Tk)$. That is why we say that f_T has *resolution* $1/T$. There are plenty of problems where it is useful to fit a model with given resolution on discrete data. All non-uniform to uniform resampling problems, like rendering on a display device or image resizing, could benefit from our approach, with T simply matched to the resolution (cut-off frequency) of the target lattice.

The regularization factor λ is a key parameter: an excessive value will oversmooth the solution, while a small value will provide a solution that is close to the data, but may have large disturbing variations. If λ is very small, the regularization term in (1) is neglictible in comparison with the fit-to-data term. So, the least-squares term is first minimized, and the remaining degrees of freedom are used to minimize $\int_{\mathcal{I}} |g^{(r)}(t)|^2$. Therefore, f_T almost passes through the samples in a region where the sampling set is sparse (Fig. 2 (b), (c), (d), for $t > 3$), and large gaps are "in-painted" in a smooth way. Conversely, if λ is large, f_T is smooth, whatever the

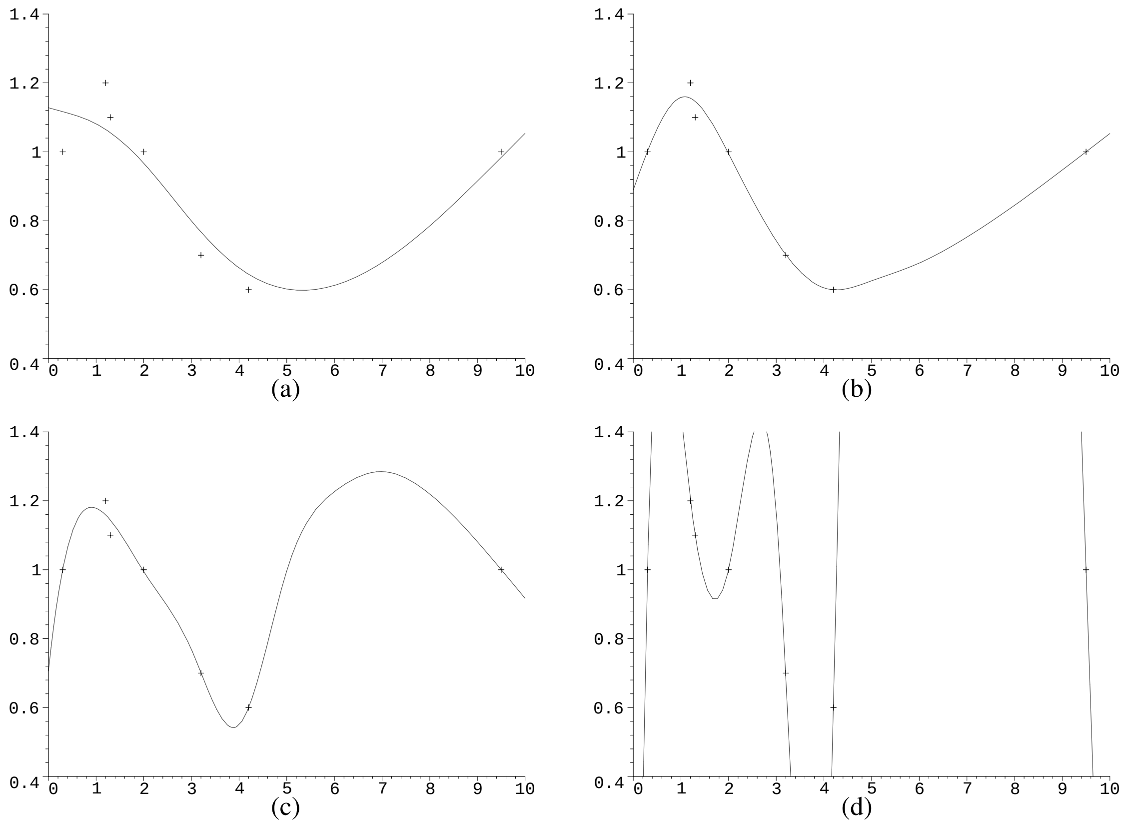

Figure 2: Uniform cubic splines with knots at the integers ($\varphi = \beta^3$, $r = 2$, $T = 1$) fitted on 7 point samples in the interval $[0, 10]$, for different values of the smoothing parameter λ. (a): $\lambda = 1.0$. (b): $\lambda = 0.001$. (c): $\lambda = 0.0001$. (d): limit case when $\lambda \to 0$. f_T is parameterized by 13 coefficients $c_T[k], k \in [-1, 11]$.

locations $x[n]$ of the data, as in Fig. 2 (a). Thus, in the noise-free case, it is tempting to choose λ very small. However, this may result in large unexpected oscillations, as shown in Fig. 2 (c), (d): the exact interpolation of the samples is a too strong constraint. Moreover, if the measurements are noisy, it is not suitable to enforce interpolation. In all cases, λ has to be tuned so as to achieve a tradeoff between the closeness of fit and the smoothness of the solution. There is no optimality rule, and the best value for the problem at hand has to be adjusted empirically on a case by case basis.

5. CONCLUSION

In this article, reconstruction from non-uniform samples has been formulated as a variational problem in a shift-invariant space. This formulation allows to bypass the usual limitations on the samples locations, that are often not pratically met. In comparison with a pure variational treatment, the reconstructed function has a given resolution that can be tuned, for example to match the representation capabilities of a target lattice for resampling purpose. We proposed a fast algorithm that computes the exact solution of the optimization problem, by solving a band-diagonal linear system without having to explicit any matrix.

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