

# RECONSTRUCTION OF DERIVATIVES: ERROR ANALYSIS AND DESIGN CRITERIA

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## ABSTRACT

We present a general Fourier-based formalism which provides an accurate prediction of the approximation error, when the derivative of a signal  $s(t)$  is continuously reconstructed from uniform point samples or generalized measurements on  $s$ . At the heart of the formalism is the *frequency error kernel*, which can be minimized to design efficient reconstruction schemes which are near optimal in the least-squares sense.

## 1. INTRODUCTION

Reconstruction of a continuous function and its derivatives from a set of samples is one of the fundamental operations in signal processing, numerical analysis, and many other fields. Edge detection, segmentation, motion estimation and super-resolution are applications where partial derivatives may be required at subpixel resolution. In visualization, the gradient is employed for shading [1]; it has to be evaluated at arbitrary locations and not only at the discrete points where the underlying signal has been sampled.

### 1.1 Motivation

We denote by  $s(t) \in L^2(\mathbb{R})$  a continuously defined function (the signal) which is prefiltered and sampled at uniform locations to yield the discrete measurements

$$u[k] = \int_{\mathbb{R}} s(t) \tilde{\varphi}(k - \frac{t}{T}) d\frac{t}{T} \quad \forall k \in \mathbb{Z}, \quad (1)$$

where  $T$  is the sampling step and the analysis function  $\tilde{\varphi}(t)$  is, typically, the impulse response of the acquisition device. This generalized sampling scenario encompasses the case where ideal point samples  $u[k] = s(Tk)$  are available, simply by letting  $\tilde{\varphi}(t)$  be the Dirac distribution  $\delta(t)$ .

The signal  $s(t)$  is unknown and the sequence  $u = (u[k])_{k \in \mathbb{Z}}$  represents the only available data. We are interested in constructing from  $u$  an estimate of the  $N$ -th derivative  $s^{(N)}(t)$  of  $s(t)$ , for some integer  $N \geq 1$ . We look for a reconstruction in a *linear shift-invariant space*  $\mathcal{V}_T(\varphi) = \text{Span}(\{\varphi(\frac{t}{T} - k)\}_{k \in \mathbb{Z}})$  generated by the translates of a template function  $\varphi(t) \in L^2(\mathbb{R})$ :

$$f(t) = \frac{1}{T^N} \sum_{k \in \mathbb{Z}} c[k] \varphi(\frac{t}{T} - k) \quad \forall t \in \mathbb{R}, \quad (2)$$

where the coefficients  $c[k]$  are obtained by discrete filtering with the stable prefilter  $p \in \ell^1$ :

$$c[k] = (u * p)[k] \quad \forall k \in \mathbb{Z}. \quad (3)$$

Using this general recipe for reconstruction, we denote by  $f_{\text{app}}$  an estimate of  $s$  (with  $N = 0$  in (2)), while an estimate of the derivative  $s^{(N)}$  is denoted by  $f_{\text{der}}$ . We remark that the reconstruction method involves a discrete prefiltering step followed by the fit of the continuous model itself. In practical applications, the prefiltering step is performed once. Its computation time is negligible in comparison with the many calls to (2) to evaluate  $f$  at the desired locations.

Estimating  $s(t)$  itself is the classical problem of interpolation, for which there is a vast amount of literature; see e.g. the survey paper [2] and some recent developments [3, 4, 5]. Of course, once an estimate  $f_{\text{app}}(t)$  of  $s(t)$  has been reconstructed, one can consider its derivative  $f_{\text{app}}^{(N)}(t)$  as a valid estimate of  $s^{(N)}(t)$ . But there is no *a priori* guarantee that whenever  $f_{\text{app}}$  is close to  $s$  in the least-squares sense, then  $f_{\text{app}}^{(N)}$  is close to  $s^{(N)}$ . Moreover, since efficiency considerations generally steer the design of the method, one may be interested in deriving direct estimation schemes of  $s^{(N)}$ , without the conceptual intermediary step of evaluating  $s$ , which unnecessarily constrains the conditions on accuracy and smoothness. In this work, we show how to evaluate the error between  $s^{(N)}$  and its estimate  $f_{\text{der}}$ , so that the design of reconstruction schemes minimizing this error is made easy.

### 1.2 Related Work

There is a vast literature on designing so-called *digital differentiators*, which are digital filters estimating the derivative at the grid points  $Tk$  only. In [6], point-wise estimates of the derivative are derived, which are optimal in the minimax sense. By contrast, we consider the context in which the derivative is reconstructed continuously in a shift-invariant space, so that it can be evaluated at every arbitrary location.

Shannon's theory provides an exact way to recover a bandlimited signal from its samples, using the sinc interpolator. Similarly, the "ideal" derivative reconstruction filter was shown in [7] to be the derivative of the sinc. However, its slow decay and the ringing artifacts it may introduce, prevent its practical use. Moreover, for non-bandlimited signals, the sinc-based theory is not valid any more [8]. That is why practitioners rely on convolutions with more localized kernels having compact support, like windowed sinc [9, 10] or piecewise polynomial functions [2, 7]. But in none of these works, there is an analytic comparison of different methods or a quantitative analysis of the estimation error.

In [11], absolute error bounds for the spatial analysis of both interpolation and derivative filters of arbitrary order are derived. Approximation theory also provides a general framework, which focuses on the asymptotic error behavior

of the reconstruction method as  $T$  tends to zero. These error bounds are generally not sharp enough to be of direct use to practitioners and more accurate ways of predicting the approximation error are desirable. For this, Blu *et al.* proposed a remarkable Fourier-based method which provides an accurate estimate of the approximation error, with a wide range of applicability [12]. Their approach makes the design of reconstruction algorithms simple and accurate, and it is at the heart of recent developments in interpolation theory [3, 4, 5]. In this paper, we show that this theory can be extended to the setting of derivative reconstruction.

### 1.3 Notations and Mathematical Safeguards

The Fourier transform of a function  $f(t)$  is denoted by  $\hat{f}(\omega) = \int_{\mathbb{R}} f(t)e^{-j\omega t} dt$ . We require  $\tilde{\varphi}$  in (1) to have a well-defined bounded Fourier transform. The  $\mathcal{Z}$ -transform of a discrete signal  $v = (v[k])_{k \in \mathbb{Z}}$  is  $V(z) = \sum_{\mathbb{Z}} v[k]z^{-k}$  and its Fourier transform is  $\hat{v}(\omega) = V(e^{j\omega})$ .

For any real  $r > 0$ , the Sobolev space  $W_2^r$  is the set of functions  $f$  such that  $\int_{\mathbb{R}} (1 + \omega^2)^r |\hat{f}(\omega)|^2 d\omega < \infty$ . Therefore, the Sobolev regularity of  $f$  is the maximum value of  $r$  such that  $f \in W_2^r$ . We assume that  $s$  has Sobolev regularity  $r > N + \frac{1}{2}$ , so that it has at least  $N$  continuous derivatives in  $L^2$ .

$\mathcal{P}^n$  is the space of polynomial functions of degree at most  $n \in \mathbb{N}$ . We define the centered B-spline  $\beta^n(t)$  of degree  $n \in \mathbb{N}$  by  $\beta^0 = \mathbb{1}_{[-\frac{1}{2}, \frac{1}{2}]}$  and  $\beta^n = \beta^{n-1} * \beta^0$ , where  $*$  denotes the convolution.

We introduce the dual  $\phi_d$  of a function  $\phi \in L^2(\mathbb{R})$  by  $\hat{\phi}_d(\omega) = \hat{\phi}(\omega)^* / \hat{a}_\phi(\omega)$ , where the discrete autocorrelation filter  $a_\phi$  is defined by  $a_\phi[k] = \int_{\mathbb{R}} \phi(t)\phi(t-k)dt$  and the star is for complex conjugation.

To a reconstruction method characterized by  $\varphi$  and  $p$ , we associate the impulse response  $\psi$  defined by

$$\psi(t) = \sum_{k \in \mathbb{Z}} p[k] \varphi(t-k) \Leftrightarrow \hat{\psi}(\omega) = \hat{p}(\omega) \hat{\varphi}(\omega). \quad (4)$$

Then,  $f_{\text{der}}(t) = \frac{1}{T^N} \sum_{k \in \mathbb{Z}} u[k] \psi(\frac{t}{T} - k)$ . We assume that the following equivalent conditions on  $\psi$ —which are equivalent as shown in [13, App. A]—are satisfied, so that  $f_{\text{der}}$  does not blow up as  $T$  tends to zero:

$$f_{\text{der}} = 0 \text{ if } s \in \mathcal{P}^{N-1}, \quad (5)$$

$$\Leftrightarrow \sum_{k \in \mathbb{Z}} P(k) \psi(t-k) = 0, \quad \forall t \in \mathbb{R}, \forall P \in \mathcal{P}^{N-1}, \quad (6)$$

$$\Leftrightarrow \hat{\psi}^{(n)}(2k\pi) = 0, \quad \forall k \in \mathbb{Z}, \forall n = 0 \dots N-1. \quad (7)$$

Our second requirement is that  $\psi$  can be decomposed as

$$\psi(t) = \sum_{k \in \mathbb{Z}} q[k] \chi(t-k) \Leftrightarrow \hat{\psi}(\omega) = \hat{q}(\omega) \hat{\chi}(\omega), \quad (8)$$

where  $q \in \ell^1$  is a discrete filter and the integer translates of the function  $\chi \in L^2$  form a Riesz basis; that is, there exist two constants  $B \geq A > 0$  such that  $A \leq \hat{\chi}(\omega) \leq B$  almost everywhere. Thus, each function of the reconstruction space  $\mathcal{Y}_T(\chi)$  has a unique and stable expansion in terms of the shifts of  $\chi$ . Note that this condition is not restrictive; in particular, there is no requirement that  $\chi$  be compactly supported, even if  $\varphi$  is.

The different functions and filters used throughout the paper are illustrated by the flowgraph in Fig. 1.

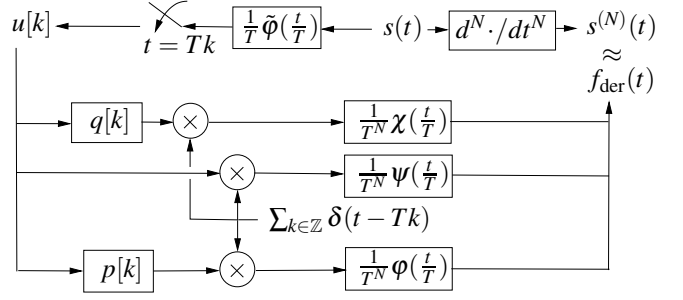


Figure 1: Flowgraph of the different equivalent procedures for the sampling process  $s \mapsto u$  and the derivative reconstruction process  $u \mapsto f_{\text{der}}$ .

### 1.4 Paper Organization

The outline of this paper is as follows. In Sect. 2, we introduce the *frequency error kernel*. In Sect. 3, we discuss the consequences of the formalism for the design of efficient reconstruction methods. Finally, in Sect. 4, we illustrate our methodology by the study of methods reconstructing the second derivative.

## 2. THE FREQUENCY ERROR KERNEL

We define the new error kernel characterizing derivative reconstruction by

$$E(\omega) = 1 - \underbrace{\frac{|\hat{\varphi}(\omega)|^2}{\hat{a}_\phi(\omega)}}_{E_{\text{min}}(\omega)} + \underbrace{\hat{a}_\phi(\omega) \left| \hat{\varphi}(\omega) \hat{p}(\omega) \frac{1}{(j\omega)^N} - \hat{\varphi}_d(\omega) \right|^2}_{E_{\text{res}}(\omega)}. \quad (9)$$

Then, we have the following results, proved in [13, App. B]:

**Theorem 1.**  $\|s^{(N)} - f_{\text{der}}\|_{L^2} = \eta_{s^{(N)}}(T) + o(T^{r-N})$ , where

$$\eta_{s^{(N)}}(T) = \left( \frac{1}{2\pi} \int_{\mathbb{R}} \underbrace{|\hat{s}(\omega)|^2 \omega^{2N}}_{|\widehat{s^{(N)}}(\omega)|^2} E(T\omega) d\omega \right)^{1/2} \quad (10)$$

Moreover, the correction term  $o(T^{r-N})$  vanishes if  $s$  is bandlimited in  $[-\frac{\pi}{T}, \frac{\pi}{T}]$ , or if  $\tilde{\varphi}$  and  $\varphi$  are both bandlimited in  $[-\pi, \pi]$ .

**Theorem 2.** In a stochastic framework where  $s$  is a realization of a random stationary process with power spectrum density  $\hat{c}_s(\omega)$ , instead of a deterministic function of  $L^2$ , we have

$$\frac{1}{T} \int_0^T \mathcal{E} \{ |s^{(N)}(t) - f_{\text{der}}(t)|^2 \} dt = \eta_{s^{(N)}}(T)^2, \quad (11)$$

by replacing  $|\hat{s}(\omega)|^2$  by  $\hat{c}_s(\omega)$  in (10).

In Fig. 2, we give an example of the error estimate  $\eta_{s^{(N)}}(T)$  for a practical experiment in which we reconstruct the derivative of a Gaussian from point samples. This shows that  $\eta_{s^{(N)}}(T)$  is an accurate, shift invariant approximation of the true error. We note that if  $s \in W_2^r$  for every  $r \geq 0$ , which is the case in our example, then the difference between  $\eta_{s^{(N)}}(T)$

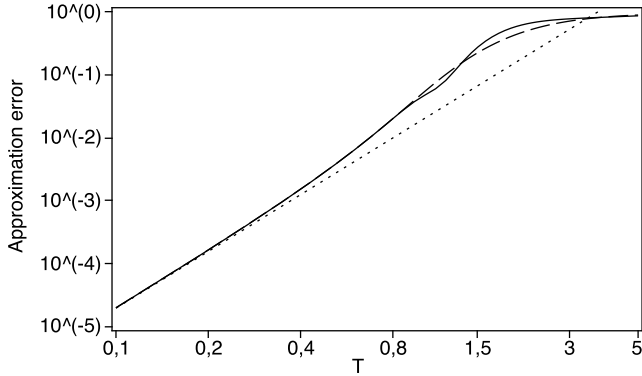


Figure 2: Approximation error as a function of the sampling step  $T$ . We estimate the first derivative  $s'(t)$  of  $s(t) = e^{-\frac{(t-1)^2}{2}}$  by the first derivative of the cubic spline interpolating the point samples  $u[k] = s(Tk)$ . Thus, we have  $\tilde{\varphi} = \delta$ ,  $\varphi = (\beta^3)'$  et  $P(z) = 6/(z+4+z^{-1})$ . The error estimate  $\eta_{s'}(T)$  (dashed line) is close to the true error  $\|s' - f_{\text{der}}\|_{L^2}$  (solid line). The dotted line is the asymptote  $C\|s^{(4)}\|_{L^2} T^3$ , where the asymptotic constant  $C = 1/\sqrt{30240}$  is obtained from the Taylor development  $E(\omega)^{1/2} \sim C\omega^3$ .

and the true error decays faster than every polynomial in  $T$  as  $T$  tends to zero. This means that  $\eta_{s^{(N)}}(T)$  can be considered as the exact value of the error in some non-infinitesimal interval; e.g. for  $T \in [0, 0.9]$  in Fig. 2. In the general case,  $\eta_{s^{(N)}}(T)$  is a reliable estimate of the error for practical values of  $T$ . By contrast, error analysis approaches based on Taylor series only apply to the asymptotic regime where  $s$  is highly oversampled.

In practical situations,  $|\hat{s}(\omega)|^2$  or  $\hat{c}_s(\omega)$  is unknown, but the multiplicative form in the integral (10) ensures that the error is small if  $E(\omega)$  is close to zero. Hence, the frequency error kernel is a tool of choice for characterizing a reconstruction scheme. More precisely, the value  $E(\omega)$  at a given frequency  $\omega$  can be interpreted as the average power of the approximation error, in case  $s^{(N)}(t)$  is the pure unit sinusoid  $e^{j\omega t/T}$  [12, Thm. 3]. Therefore, the study of  $E(\omega)$  allows to characterize the behavior of a reconstruction method at different frequency components. For instance,  $E(\omega)$  for  $\omega$  close to  $\pi$  indicates to which extent the salient features, fine details and textures are preserved and aliasing is enhanced. By contrast, an asymptotic study for  $\omega$  around 0 characterizes the reconstruction quality for the low frequency content of the signal  $s$ .

Thus,  $\varphi$  and  $p$  can be tuned to minimize the error kernel, so that the reconstruction quality is improved for virtually every function  $s$  [3, 4, 5, 14]. Given the reconstruction space  $\mathcal{V}_T(\varphi)$ , the error kernel  $E(\omega)$  attains its minimum value  $E_{\min}(\omega)$ , for every  $\omega \in \mathbb{R}$ , when  $f_{\text{der}}$  is the minimum error reconstruction of  $s^{(N)}$  in  $\mathcal{V}_T(\varphi)$ ; that is, its orthogonal projection onto  $\mathcal{V}_T(\varphi)$ . Thus, the prefilter  $p$  should be designed so that  $E(\omega)$  is close to  $E_{\min}(\omega)$ , in order for the method to behave like this optimal, but generally unattainable, least-squares approximation [12, 5].

### 3. ASYMPTOTIC APPROXIMATION PERFORMANCE

In this section, we focus on the reconstruction of lowpass signals; that is, we assume that  $s^{(N)}$  has most of its energy around  $\omega = 0$ . This is the case for natural images, at least for  $N = 1$ . For other types of signals, the same study could be performed around another frequency than 0.

#### 3.1 The Approximation Order

From Theorem 1, due to the closed form of  $\eta_{s^{(N)}}(T)$ , it is easy to expand this estimate in a power series of  $T$  to obtain the exact behavior of the error as  $T \rightarrow 0$ . Specifically, if  $s^{(N)}$  has at least Sobolev regularity  $L$ , we have the equivalence

$$E(\omega)^{1/2} \sim C\omega^L \quad \text{as } \omega \rightarrow 0 \quad (12)$$

$$\text{iff } \|s^{(N)} - f_{\text{der}}\|_{L^2} \sim C\|s^{(N+L)}\|_{L^2} T^L \quad \text{as } T \rightarrow 0. \quad (13)$$

In that case, we speak about a  $L$ th-order approximation scheme. When most of the spectral energy of the signal is concentrated in the neighborhood of  $\omega = 0$ , the approximation order  $L$  is the most crucial determinant of the reconstruction quality and should be chosen as large as possible. To have an approximation order  $L$ , it is necessary that  $\chi$ , defined in (8), satisfies the Strang-Fix conditions of order  $L$  [15]:

$$\hat{\chi}(0) \neq 0 \quad \text{and} \quad \hat{\chi}^{(n)}(2k\pi) = 0 \quad \text{for} \quad \begin{cases} k \neq 0 \\ n = 0 \dots L-1 \end{cases} \quad (14)$$

It was shown in [3] that a function  $\chi$  with approximation  $L$  has a support size  $S \geq L$  with equality iff  $\chi$  is a MOMS. Therefore, the reconstruction schemes having the optimal tradeoff between the reconstruction quality and the computational complexity are obtained by choosing the reconstruction kernel  $\varphi$  as a MOMS.

#### 3.2 The choice of the Prefilter

We now assume that  $\varphi$  is fixed. This determines  $\chi$  and its approximation order  $L$  (eqn. (14)). Then, we have to choose  $p$  so as to exploit at best the properties of the reconstruction space  $\mathcal{V}_T(\varphi)$ ; that is, so that the scheme has approximation order  $L$  (eqns (12),(13)). In fact,  $f_{\text{der}} \in \mathcal{V}_T(\varphi) \subset \mathcal{V}_T(\chi)$  and  $p$  controls which approximation of  $s^{(N)}$  in  $\mathcal{V}_T(\varphi)$  is picked by the method. The best possible reconstruction is the orthogonal projection of  $s^{(N)}$  in  $\mathcal{V}_T(\varphi)$ . The error between this optimal approximation and  $s^{(N)}$  decays like  $T^L$  and is characterized by the error kernel  $E_{\min}$  given in (9). Thus, the reconstruction scheme has approximation order  $L$  if and only if

$$E(\omega) = E_{\min}(\omega) + O(\omega^{2L}). \quad (15)$$

This is equivalent to the quasi-biorthogonality conditions:

$$\frac{\hat{p}(\omega)}{(j\omega)^N} \hat{\varphi}(\omega) \hat{\varphi}(\omega + 2k\pi) = \delta_k + O(\omega^L), \quad \forall k \in \mathbb{Z}, \quad (16)$$

or, equivalently,

$$\hat{p}(\omega) \hat{\varphi}(\omega) = \frac{(j\omega)^N}{\hat{\varphi}(\omega)} + O(\omega^{L+N}). \quad (17)$$

Thus, we have to choose  $p$  so that these  $L+N$  linear constraints are satisfied. There is a great freedom in this respect.

#### 4. CASE STUDY: RECONSTRUCTION OF THE SECOND DERIVATIVE

To illustrate the benefits of the framework, we compare several methods reconstructing the second derivative from point samples ( $N = 2$ ,  $\tilde{\varphi} = \delta$ ), by means of their error kernels.

1. The first method consists in applying the finite difference filter  $P(z) = z - 2 + z^{-1}$  to the data, then in interpolating the obtained sequence using the linear B-spline  $\varphi = \beta^1$ . Thus, the reconstructed function  $f_{\text{der}}$  is piecewise linear and has global regularity  $\mathcal{C}^0$ .
2. The second method is similar to the first one, with cubic spline instead of linear spline interpolation.  $p$  is the combination of the finite difference filter and of the interpolation prefilter [2]:  $P(z) = 6(z - 2 + z^{-1})/(z + 4 + z^{-1})$ , and  $\varphi = \beta^3$  is the cubic B-spline. Thus, the reconstructed function  $f_{\text{der}}$  is piecewise cubic and has global regularity  $\mathcal{C}^2$ .
3. The third method consists in computing the second derivative of the cubic spline interpolating the samples  $u[k]$ . Then, we have  $P(z) = 6/(z + 4 + z^{-1})$  and  $\varphi = (\beta^3)''$ . Since we have the property  $(\beta^3)''(t) = \beta^1(t - 1) - 2\beta^1(t - 1) + \beta^1(t + 1)$ , the method can also be implemented by using  $P(z) = 6(z - 2 + z^{-1})/(z + 4 + z^{-1})$  and  $\varphi = \beta^1$ . We notice that this third method is hybrid between the two previous ones, with the generator  $\varphi$  of the first method and the prefilter of the second one.
4. The last method, by  $\varepsilon$ -differentiation, consists in applying a finite difference to the spline  $f_{\text{app}}(t)$  interpolating the samples  $u[k]$ :

$$f_{\text{der}}(t) = \frac{1}{\varepsilon^2} \left( f_{\text{app}}(t - \varepsilon) - 2f_{\text{app}}(t) + f_{\text{app}}(t + \varepsilon) \right), \quad (18)$$

for some  $\varepsilon > 0$ . This method is particularly interesting for applications where estimates of both  $s(t)$  and  $s''(t)$  have to be reconstructed at the same time, for instance in volume rendering [16]. We remark that when  $\varepsilon \rightarrow 0$ ,  $f_{\text{der}}$  converges to the second derivative  $f''_{\text{app}}$ , which corresponds to method 3). Moreover, if  $\varepsilon = 1$ , the method is equivalent to method 2). In the general case, the method formally amounts to take  $P(z) = 6/(z + 4 + z^{-1})$  and  $\varphi(t) = (\beta^3(t - \varepsilon) - 2\beta^3(t) + \beta^3(t + \varepsilon))/\varepsilon^2$ . The reconstructed function  $f_{\text{der}}$  is a cubic spline, with global regularity  $\mathcal{C}^2$ , but with non-uniform knots. The question arises how to choose the optimal value of  $\varepsilon$ . We will see that the error kernel provides us with a simple way of deriving this value.

We first notice that the four methods have approximation order 2. Indeed, we have the Taylor series  $E(\omega)^{1/2} \sim C\omega^2$ , where the asymptotic constant  $C$  is, for the first three methods,  $C = \frac{\sqrt{105}}{60} \approx 0.17$ ,  $C = \frac{1}{12} \approx 0.08$ ,  $C = \frac{\sqrt{5}}{60} \approx 0.04$ , respectively. For the fourth method, the value of  $C$  depends on  $\varepsilon$ . For  $\varepsilon \in (0, 1/2]$ , we have:

$$C^2 = \frac{1}{720} - \frac{1}{72}\varepsilon^2 + \frac{31}{1260}\varepsilon^3 - \frac{1}{180}\varepsilon^4 \quad (19)$$

and  $C$  is higher for  $\varepsilon > 1/2$ . Thus, we can choose  $\varepsilon$  so that  $C$  is minimized, which yields the optimal value

$$\varepsilon_o = \frac{93}{56} - \frac{1}{56}\sqrt{4729} \approx 0.43, \quad (20)$$

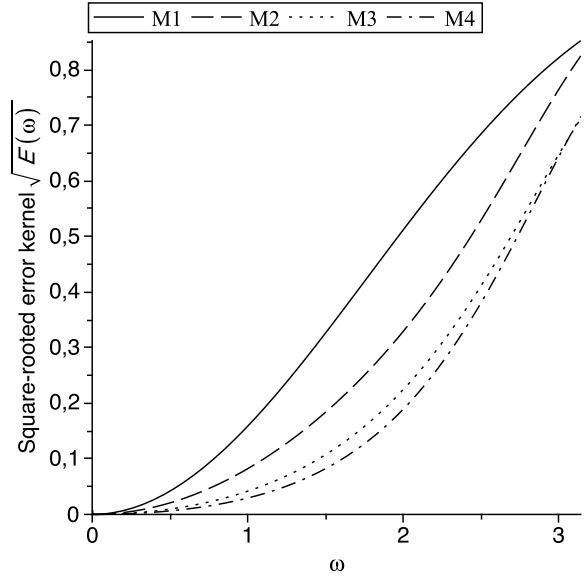


Figure 3: Square-rooted frequency error kernel  $E(\omega)^{1/2}$  for the reconstruction methods 1 to 4 described in Sect. 4, denoted M1 to M4, respectively. Only the values in the half Nyquist band  $\omega \in [0, \pi]$  are plotted, since  $E(\omega)$  is symmetrical in 0.

for which  $C \approx 6.10^{-4}$ .

We depicted the error kernels associated to the four methods in Fig. 3, with  $\varepsilon = \varepsilon_o$  for the method 4). We observe that the hierarchy of the methods with respect to their asymptotic constants  $C$  is respected. In other words, the minimization of  $C$ , which is an asymptotic constraint in  $\omega = 0$ , provides error kernels whose good behavior extends in the whole Nyquist band  $\omega \in [-\pi, \pi]$ . This observation, which is not expected *a priori*, was already done for interpolation [3, 4, 5, 14]. Thus, the minimization of the asymptotic constant, for a given approximation order, is a simple and efficient way of designing reconstruction methods of high quality. Applying this methodology to the reconstruction of derivatives is new and the authors are not aware, for instance, of results similar to (20) in the literature.

It is interesting to determine a function  $\chi$  corresponding to the method 4). In fact, the translates of  $\varphi$  do not form a Riesz basis. We can define  $\chi$  by

$$\hat{\chi}(\omega) = \frac{\hat{\varphi}(\omega)}{2 \cos(\omega) - 2} = \frac{2 \cos(\varepsilon\omega) - 2}{\varepsilon^2 (2 \cos(\omega) - 2)} \hat{\beta}^3(\omega), \quad (21)$$

which satisfies the Strang-Fix conditions of order 2. Actually, one can show that  $\chi = \beta^1 * \beta_\varepsilon^1$  where  $\beta_\varepsilon^1(t) = \frac{1}{\varepsilon} \beta^1(\frac{t}{\varepsilon})$ .  $\chi$  has compact support in  $[-1 - \varepsilon, 1 + \varepsilon]$ , is piecewise cubic and has global regularity  $\mathcal{C}^2$ .

There is an important remark concerning method 3). We observe that  $E = E_{\text{min}}$  and, indeed,  $f_{\text{der}}$  is the orthogonal projection of  $s''$  in  $\mathcal{V}_T(\beta^1)$ . In other words, the method yields the best possible piecewise linear approximation of  $s''$ , although this function is unknown. More generally, it is possible to obtain the orthogonal projection of  $s^{(N)}$  in the spline space  $\mathcal{V}_T(\beta^{N-1})$  from point samples of  $s$ , for every  $N \geq 1$ . This is a remarkable property of spline spaces.

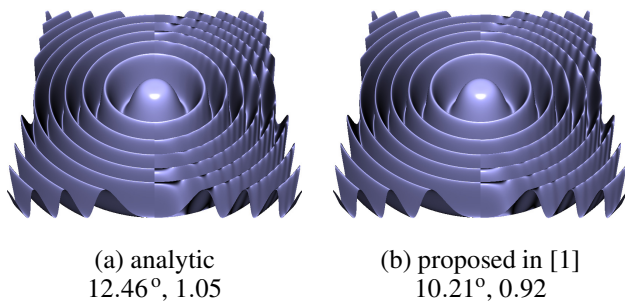


Figure 4: The 0.5 isosurface of the synthetic Marschner-Lobb function is shaded using two gradient reconstruction schemes from the sampled data ( $41 \times 41 \times 41$ ) [1]. The analytic function is used to reconstruct the isosurface and the left half of each image shows the truth. The mean angular error and the mean length of the error vector are indicated. (a) the analytic gradient of the cubic spline interpolation of the samples is used. (b) the CC-s scheme proposed in [1] is used, which is based on the formalism proposed in this paper.

Finally, we note that the reconstruction spaces of methods 1), 3), 4) have approximation order 2, while the cubic spline space of method 2) has approximation order 4. Hence, for method 2), the prefilter  $p$  does not fully exploit the representation power of  $\mathcal{V}_T(\beta^3)$ . We can propose another prefilter so that the method has approximation order 4. After (17), this is equivalent to have

$$\hat{p}(\omega) = -\omega^2 \hat{\beta}_d^3(\omega) + O(\omega^6) = -\omega^2 - \frac{1}{6}\omega^4 + O(\omega^6). \quad (22)$$

For instance, a solution is  $P(z) = 120(z - 2 + z^{-1}) / (66 + 26(z + z^{-1}) + z^2 + z^{-2})$  and the method amounts to compute the second derivative of the spline of degree 5 interpolating the samples  $u[k]$ .

## 5. CONCLUSION

We introduced a generic Fourier methodology to evaluate the quality of shift-invariant methods that continuously reconstruct the derivative of a function from discrete measurements. In our future work, we will focus on the applications to visualization [1], as illustrated in fig. 4, and investigate the extension of the formalism to noisy measurements [17, 18].

Since the frequency error kernel can be defined for multi-dimensional signals on lattices, like in [14], the extension of this work to the evaluation of partial derivatives of multi-dimensional signals is straightforward [1]. Interesting applications include finite difference methods and the numerical resolution of PDEs.

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