# New Efficient Implementation of the Discrete Wavelet Transform with Arbitrary FIR Analysis Filters

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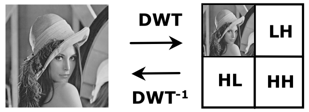
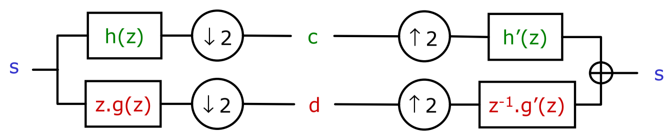
**Abstract:** We propose a new implementation of the Discrete Wavelet Transform (DWT) with arbitrary FIR analysis filters; that is, the synthesis filters are not constrained to be also FIR, as usually imposed.

## The Discrete Wavelet Transform

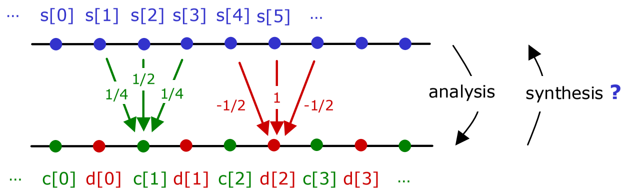
**Image or signal analysis with the DWT reverts to feeding a signal in a perfect reconstruction filter-bank (Mallat's algorithm)**



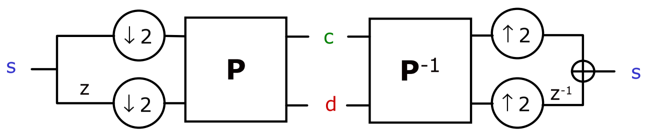

- h(z) : analysis, low-pass
- g(z) : analysis, high-pass
- g'(z) : synthesis, low-pass
- h'(z) : synthesis, high-pass

**Example:**




$$h=\left[\frac{1}{4}\ \ \frac{1}{2}\ \ \frac{1}{4}\right] \qquad g=\left[\frac{-1}{2}\ \ 1\ \ \frac{-1}{2}\right]$$

**Equivalent representation with the polyphase matrix :**




$$\mathbf{P}(z)=\begin{bmatrix}\frac{1}{2} & \frac{1}{4}+\frac{1}{4}z^{-1}\\ \frac{-1}{2}z+\frac{-1}{2} & 1\end{bmatrix}$$

$$\det(\mathbf{P}) = \frac{1}{8}z+\frac{3}{4}+\frac{1}{8}z^{-1}$$

**The DWT is FIR invertible iff det(P) is a pure delay**

## Proposed method: A new factorization

$$\underbrace{\begin{bmatrix}\ddots &&&&&&&&\\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} &&&&&&\\ & \frac{-1}{2} & 1 & \frac{-1}{2} &&&&&\\ && \frac{1}{4} & \frac{1}{2} & \frac{1}{4} &&&&\\ &&& \frac{-1}{2} & 1 & \frac{-1}{2} &&&\\ &&&& \frac{1}{4} & \frac{1}{2} & \frac{1}{4} &&\\ &&&&& \frac{-1}{2} & 1 & \frac{-1}{2} &\\ &&&&&&&& \ddots\end{bmatrix}}_{\mathbf{M}}\begin{bmatrix}\vdots\\ s[0]\\ s[1]\\ s[2]\\ s[3]\\ s[4]\\ s[5]\\ \vdots\end{bmatrix}=\underbrace{\begin{bmatrix}\vdots\\ c[0]\\ d[0]\\ c[1]\\ d[1]\\ c[2]\\ d[2]\\ \vdots\end{bmatrix}}_{\mathbf{w}}$$

**Algebraic view:**

analysis: **w=Ms**

synthesis: infinite banded block-Toeplitz system solving for **s** in **Ms=w**

method for inverse DWT (synthesis) : **LU-factorize M**

$$\mathbf{M}=\begin{bmatrix}\ddots &&&&&\\ & \beta/2 &&&&\\ && \beta &&&\\ &&& \beta/2 &&\\ &&&& \beta &\\ &&&&& \ddots\end{bmatrix}\begin{bmatrix}\ddots &&&&&\\ \alpha & 1 & 0 &&&\\ & -\alpha & 1 & 0 &&\\ && \alpha & 1 & 0 &\\ &&& -\alpha & 1 & 0\\ &&&&& \ddots\end{bmatrix}\begin{bmatrix}\ddots &&&&&\\ 0 & 1 & \alpha &&&\\ & 0 & 1 & -\alpha &&\\ && 0 & 1 & \alpha &\\ &&& 0 & 1 & -\alpha\\ &&&&& \ddots\end{bmatrix}$$

$$\mathbf{P}=\begin{bmatrix}\beta/2 & 0\\ 0 & \beta\end{bmatrix}\begin{bmatrix}1 & \alpha z^{-1}\\ -\alpha & 1\end{bmatrix}\begin{bmatrix}1 & \alpha\\ -\alpha z & 1\end{bmatrix}$$

$\alpha=\sqrt{2}-1 \qquad \beta=(\sqrt{2}+1)/2$

### Corresponding implementation:

analysis:

```
for n = -∞ ... ∞
    c[n] = s[2n] + αs[2n+1]
    d[n] = s[2n+1] - αs[2n+2]
for n = ∞ ... -∞
    d[n] = d[n] - αc[n]
    c[n] = c[n] + αd[n-1]
scaling
```

synthesis:

```
1/scaling
for n = -∞ ... ∞
    c[n] = c[n] - αd[n-1]
    d[n] = d[n] + αc[n]
for n = ∞ ... -∞
    s[2n+1] = d[n] + αs[2n+2]
    s[2n] = c[n] - αs[2n+1]
```

The inverse algorithm is the same as the forward one, up to time reversal + sign flipping.

**Our method: causal/anti-causal factorization of the polyphase matrix**

find the (true) polynomials $P_1(z)...P_8(z)$ such that $\mathbf{P}(z)=$

$$\begin{bmatrix}\beta_1 & 0\\ 0 & \beta_2\end{bmatrix}\begin{bmatrix}1+z^{-1}P_1(z^{-1}) & z^{-1}P_2(z^{-1})\\ P_3(z^{-1}) & 1+z^{-1}P_4(z^{-1})\end{bmatrix}\begin{bmatrix}1+zP_5(z) & P_6(z)\\ zP_7(z) & 1+zP_8(z)\end{bmatrix}$$

quadratic system to solve for finding their coefficients

**Key points: New interpretation of the DWT that allows to concentrate on the analysis part. The synthesis follows from the PR property. This preliminary study opens new perspectives, with the same advantages as the lifting scheme: fast transform, in-place calculation, int.-to-int. tranforms, extensions to separable multi-D domains, non-uniform domains...**

**The principle of LU (causal/anti-causal) factorization can be applied to other maximally decimated transforms: m-channel filter-banks, rational (non-dyadic) filter-banks...**