TORWARDS A GENERAL FORMULATION FOR OVER-SAMPLING AND UNDER-SAMPLING

Akira Hirabayashi[1] *and Laurent Condat*[2]

[1] Dept. of Information Science and Engineering, Yamaguchi University,
2-16-1, Tokiwadai, Ube 755-8611, Japan
phone: + (81) 836 85 9516, fax: + (81) 836 85 9501, email: a-hira@yamaguchi-u.ac.jp
web: www.ir.csse.yamaguchi-u.ac.jp/english/index_e.html

[2] Institute of Biomathematics and Biometry, GSF National Research Center for Environment and Health,
Ingolstädter Landstrasse 1, D-85764 Neuherberg, Germany
phone: + (49) 89 2891 8322, fax: + (49) 89 3187 3369, email: condat@ma.tum.de
web: http://www-m6.ma.tum.de/ condat/

ABSTRACT

We investigate over-sampling and under-sampling scenarios under the formulation of a generalized sampling model. Usually, these scenarios are described in the context of the so-called Shannon's sampling theorem. This can be easily extended to more general settings. We first revisit a conventional definition of over-sampling and under-sampling in a general setting, and point out that the definition consists of two conditions. To treat them separately, we introduce the two notions of 'perfect reconstruction' and 'redundant sampling.' We show that these concepts are geometrically characterized by using sampling and reconstruction spaces. Then, we show that there appear four types of scenarios, which includes the conventional over-sampling and normal sampling, and further two types of under-sampling scenarios. The second type is more counter intuitive because it satisfies both non-perfect reconstruction and redundant sampling scenarios. We illustrate this last scenario by a practical example that involves cyclic B-spline functions.

1. INTRODUCTION

Over-sampling and under-sampling are frequently appearing terms in the field of signal/image processing. Over-sampling is typically useful for noise reduction, while under-sampling is a situation often encountered in real-world problems, since nature has an infinite amount of information.

Over-sampling and under-sampling are usually described within the following framework [1]: if a signal f, which contains no frequencies higher than the frequency $\omega_c/2$, is sampled at a frequency ω_s greater than or equal to ω_c, then the signal can be reconstructed via the formula

$$f(x) = \frac{\omega_c}{\omega_s} \sum_{n=-\infty}^{\infty} f\left(\frac{n}{\omega_s}\right) \frac{\sin \pi\omega_c(x-n/\omega_s)}{\pi\omega_c(x-n/\omega_s)}. \quad (1)$$

Over-sampling means that ω_s is greater than ω_c. *Normal sampling* (a.k.a., critical sampling) is a term sometimes used for the case where $\omega_s = \omega_c$. On the other hand, *under-sampling* means that ω_s is less than ω_c and in this case, f is not generally reconstructed by Eq. (1) anymore. The latter case causes the aliasing problem.

A recent trend for discussing the sampling theorem is not restricted to the above formulation [2]∼[9]. That is, the signal is reconstructed by not only the so-called sinc function, but also general reconstruction functions like splines. Further on, samples are not only the ideal samples $f(n/\omega_s)$, but also generalized samples d_n which are modeled by the inner product between the target signal f and general sampling functions. The coefficients for the linear combination of the reconstruction functions are obtained from samples by a correction filter.

When we look at over-sampling and under-sampling from the viewpoint of this generalized formulation, we arrive at the idea that the sampling frequency is not the essential point of these sampling scenarios. Instead, over-sampling essentially means that samples are linearly dependent for any bandlimited signal f, while under-sampling means that there exists some bandlimited signal f which can not be perfectly reconstructed from its samples.

Taking these perspectives into account, over-sampling and under-sampling were defined in the generalized formulation in [8]. This definition consists of two conditions, which can be treated separately, but not so. Hence, for a more detailed analysis, we introduce two concepts, *perfect reconstruction* and *redundant sampling*. We characterize these scenarios geometrically by using sampling and reconstruction spaces. Then, by combinations of the two concepts, we derive four types of scenarios. Two of them directly correspond to over-sampling and normal sampling scenarios in the conventional sense. On the other hand, both of the rest two scenarios correspond to conventional under-sampling scenario. We call them under-sampling scenarios of the first and the second types. Interestingly, the second type satisfies both non-perfect reconstruction and redundant sampling (a counter intuitive situation). By using examples of a cyclic B-spline functions, we show that under-sampling scenario of the second type may appear in practical situations.

1.1 Notations and Mathematical Preliminaries

We will make use of the following notations. The measurements of a signal are represented as a vector in the N-dimensional unitary space $\mathbf{C}^N$. The reconstructed signal, on the other hand, will be parameterized by a vector in $\mathbf{C}^K$. The

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standard bases for $\mathbf{C}^N$ and $\mathbf{C}^K$, are $\{\mathbf{e}_n^{(N)}\}_{n=1}^N$ and $\{\mathbf{e}_k^{(K)}\}_{k=1}^K$, respectively. That is, $\mathbf{e}_n^{(N)}$ (resp., $\mathbf{e}_k^{(K)}$) is the N-dimensional (resp., K-dimensional) vector consisting of zero elements except for the n-th (resp., k-th) element which is equal to 1.

The orthogonal complement of the subspace S is denoted by $S^\perp$. $\mathscr{R}(T)$ and $\mathscr{N}(T)$ stand for the range and the null space of the operator T, respectively. T^* is the adjoint operator of T.

Let α and β be elements of two Hilbert spaces H_1 and H_2, respectively. Let $(\alpha \otimes \overline{\beta})$ be an operator from H_2 to H_1 defined by

$$(\alpha \otimes \overline{\beta})\gamma = \langle \gamma, \beta \rangle \alpha \quad \text{for any} \quad \gamma \in H_2, \tag{2}$$

where $\langle \cdot, \cdot \rangle$ is the inner product in H_2. This operator is called the Neumann-Schatten product [10] and it satisfies the relation

$$\left(\alpha \otimes \overline{\beta}\right)^* = \beta \otimes \overline{\alpha}. \tag{3}$$

2. FORMULATION OF SAMPLING AND RECONSTRUCTION PROBLEM

We start with the formulation of the sampling problem, which is illustrated in Fig. 1. The original input signal f is defined over a continuous domain $\mathscr{D}$ and is assumed to belong to a Hilbert space $H = H(\mathscr{D})$. The measurements of f, denoted by d_n $(n = 1, 2, \ldots, N)$, are given by the inner product in H of f with the sampling functions $\{\psi_n\}_{n=1}^N$:

$$d_n = \langle f, \psi_n \rangle. \tag{4}$$

The N-dimensional vector consisting of d_n is denoted by $\mathbf{d}$. Let A_s be the operator that maps f into $\mathbf{d}$:

$$A_s f = \mathbf{d}. \tag{5}$$

By using the Neumann-Schatten product, the operator A_s is expressed without f as

$$A_s = \sum_{n=1}^{N} \mathbf{e}_n^{(N)} \otimes \overline{\psi_n}.$$

The reconstructed signal $\tilde{f} \in H$ is given by a linear combination of reconstruction functions $\{\varphi_k\}_{k=1}^K$:

$$\tilde{f} = \sum_{k=1}^{K} c_k \varphi_k. \tag{6}$$

The K-dimensional vector of coefficients c_k is denoted by $\mathbf{c}$. We introduce the (adjoint) reconstruction operator:

$$A_r = \sum_{k=1}^{K} \mathbf{e}_k^{(K)} \otimes \overline{\varphi_k}. \tag{7}$$

It follows from Eqs. (6), (2), (3), and (7) that

$$\tilde{f} = A_r^* \mathbf{c}. \tag{8}$$

Let X be the $K \times N$ matrix that maps $\mathbf{d}$ to $\mathbf{c}$:

$$X\mathbf{d} = \mathbf{c}. \tag{9}$$

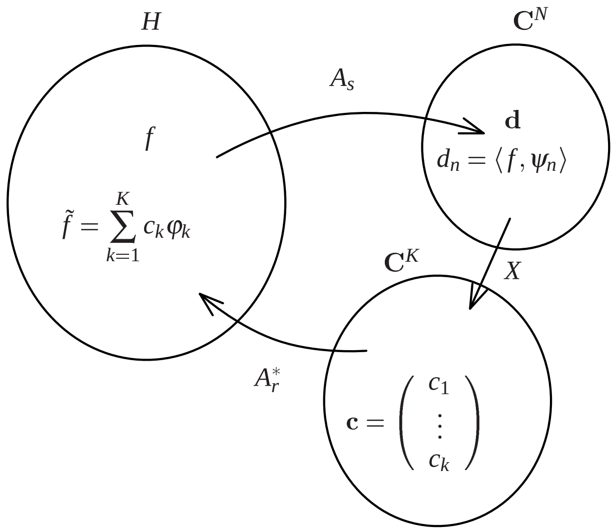

Figure 1: Schematic view of sampling and reconstruction.

Then, Eqs. (8), (9), and (5) yield

$$\tilde{f} = A_r^* X A_s f. \tag{10}$$

With this formulation, the sampling problem becomes equivalent to finding a suitable matrix X so that $\tilde{f}$ satisfies some optimality criterion, such as least squared error [2], consistency [3]∼[8], and minimax regret [9].

Let V_s and V_r be subspaces in H spanned by $\{\psi_n\}_{n=1}^N$ and $\{\varphi_k\}_{k=1}^K$, respectively. They are called the *sampling space* and the *reconstruction space*, respectively. They play important roles in this paper. It holds that

$$V_s = \mathscr{R}(A_s^*),$$

$$V_r = \mathscr{R}(A_r^*).$$

3. CONVENTIONAL DEFINITION OF OVER-SAMPLING AND UNDER-SAMPLING

As mentioned in Introduction, over-sampling and under-sampling are usually defined with reference to Shannon's sampling theorem for band-limited signals. In that context, over-sampling means sampling at a rate that is above the critical Nyquist rate, while under-sampling means sampling at a lesser rate. It is obviously possible to reconstruct band-limited signals perfectly in the former case but generally not in the latter.

With the formulation in Section 2, these concepts are translated as follows. First, the band-limited property is generalized to the fact that f belongs to the reconstruction space V_r. Second, the over-sampling scenario corresponds to the case where the sampled measurements $\{d_n\}_{n=1}^N$ are linearly dependent for any f in V_r. Third, the perfect reconstruction property means that there exists X that satisfies

$$A_r^* X A_s f = f \tag{11}$$

for any f in V_r. These considerations were summarized in the following definition:

Definition 1 [8] *With the formulation in Section 2, if there exists some X that satisfies Eq. (11) for any f in V_r, then*

Table 1: Summary of sampling and reconstruction scenarios (see text).

		Non-redundant sampling	Redundant sampling
	Geometric characterization	$\{\hat{\psi}_n\}_{n=1}^N$: Independent	$\{\hat{\psi}_n\}_{n=1}^N$: Dependent
Perfect reconstruction	$V_r \cap V_s^{\perp} = \{0\}$	Normal sampling	Over-sampling
Non-perfect reconstruction	$V_r \cap V_s^{\perp} \neq \{0\}$	Under-sampling 1	Under-sampling 2

we have an over-sampling (resp. normal sampling) scenario over V_r depending on whether the sampled measurements $\{d_n\}_{n=1}^N$ are linearly dependent for any f in V_r or not. If, on the other hand, there is no X that satisfies Eq. (11) for any f in V_r, then we have an under-sampling scenario.

We can see from this definition that an under-sampling scenario is defined by a single condition, the existency of the operator X, while over-sampling and normal sampling scenarios are defined by two conditions, the linearly dependency of samples in addition to the former condition. Note that we can treat these conditions in a separate way. Hence, in the following section, we define these two concepts explicitly, and characterize them geometrically.

4. SAMPLING AND RECONSTRUCTION SCENARIOS

In this section, we introduce two notions, perfect reconstruction and redundant sampling. After giving their geometric characterizations, we show the relations of these notions to Definition 1.

4.1 Perfect Reconstruction

The existence of the operator X that satisfies Eq. (11) for any f in V_r, originally means that we can perfectly reconstruct all signals f in V_r from the samples $\{d_n\}_{n=1}^N$. Hence, we define the perfect reconstruction scenario as follows.

Definition 2 *We have a perfect reconstruction scenario if there exists an operator X that satisfies Eq. (11) for any f in V_r.*

By rephrasing Proposition 1 in [8], we can geometrically characterize this scenario as follows:

Theorem 1 [8] *We have a perfect reconstruction scenario if and only if*

$$V_r \cap V_s^{\perp} = \{0\}. \tag{12}$$

That is, over-sampling and normal sampling scenarios are covered by Eq. (12), while an under-sampling scenario is characterized by

$$V_r \cap V_s^{\perp} \neq \{0\}. \tag{13}$$

Eq. (13) means that there exists some nonzero signals f in V_r which are mapped into 0 through the sampling operator: $A_s f = 0$. Hence, we can not reconstruct such signals by the formula $\tilde{f} = A_r^* \mathbf{c} = A_r^* X \mathbf{d}$.

In the context of the consistency sampling theorem, Eq. (12) was assumed implicitly in [3] and explicitly in [6]. An under-sampling case of Eq. (13) was investigated in [8].

4.2 Redundant Sampling

Our next attention is focused on the linearly dependency condition in Definition 1. We introduce the following concept:

Definition 3 *We have a redundant sampling scenario if the sampled measurements $\{d_n\}_{n=1}^N$ are linearly dependent for any f in V_r, i.e., if it holds for any f in V_r and for some nonzero coefficients $\{a_n\}_{n=1}^N$ that*

$$\sum_{n=1}^{N} a_n d_n = 0. \tag{14}$$

Eq. (4) allows us to express Eq. (14) as

$$\sum_{n=1}^{N} a_n \langle f, \psi_n \rangle = 0, \tag{15}$$

in which f in V_r is explicitly shown.

This scenario is geometrically characterized as follows. Let $\hat{\psi}_n$ be the orthgonal projection of ψ_n onto V_r:

$$\hat{\psi}_n = P_{V_r} \psi_n. \tag{16}$$

Theorem 2 *We have a redundant sampling scenario if and only if $\{\hat{\psi}_n\}_{n=1}^N$ are linearly dependent, i.e., for some nonzero coefficients $\{a_n\}_{n=1}^N$, it holds that*

$$\sum_{n=1}^{N} a_n \hat{\psi}_n = 0. \tag{17}$$

(Proof) Eq. (15) implies that Eq. (14) is equivalent to

$$\left\langle f, \sum_{n=1}^{N} a_n \psi_n \right\rangle = 0$$

for any f in V_r. This is further equivalent to

$$P_{V_r} \sum_{n=1}^{N} a_n \psi_n = 0.$$

This is equivalent to Eq. (17) because of Eq. (16). ■

Theorem 2 implies that a non-redundant sampling scenario is characterized by the orthogonal projections of sampling functions $\{\psi_n\}_{n=1}^N$ onto the reconstruction space V_r. Let us show some sufficient conditions for Eq. (17).

Corollary 1 *We have a redundant sampling scenario if the sampling functions $\{\psi_n\}_{n=1}^N$ are linearly dependent, i.e., for some nonzero coefficients $\{a_n\}_{n=1}^N$, it holds that*

$$\sum_{n=1}^{N} a_n \psi_n = 0. \tag{18}$$

Proof is abbreviated. Although Eq. (18) is an obvious condition, we should note that this is a sufficient condition, not a necessary and sufficient condition. This means that there can be a case in which sampled measurements are linearly dependent even if sampling functions are linearly independent. The following corollary suggest that such a situation really exists.

Corollary 2 *We have a redundant sampling scenario if the sampling and the reconstruction spaces satisfy the following condition:*

$$V_s \cap V_r^{\perp} \neq \{0\}. \tag{19}$$

(Proof) Assume that Eq. (19) holds. Then, there exists a nonzero element g in $V_s \cap V_r^{\perp}$. Since g belongs to V_s, it holds for some nonzero coefficients $\{a_n\}_{n=1}^{N}$ that

$$g = \sum_{n=1}^{N} a_n \psi_n. \tag{20}$$

Since g is orthogonal to V_r, it holds for any f in V_r that

$$\left\langle f, \sum_{n=1}^{N} a_n \psi_n \right\rangle = 0.$$

Then, Eq. (4) implies Eq. (14). ■

Eq. (19) is important because this equation shows that, even if sampling functions are linearly independent, sampled measurements can be linearly dependent. Interestingly, Eq. (19) is similar to Eq. (13) with the role of V_r and V_s exchanged.

By taking a contraposition of Corollary 2, we have

Corollary 3 *It holds that*

$$V_s \cap V_r^{\perp} = \{0\} \tag{21}$$

if we have a non-redundant sampling scenario.

Orthogonal complement of Eq. (21) yields

$$V_r + V_s^{\perp} = H, \tag{22}$$

which is a condition assumed in [8] together with Eq. (13). Hence, now we can see that the study in [8] covered not only non-perfect reconstruction and non-redundant sampling scenarios, but also some part of non-perfect reconstruction and redundant sampling scenarios.

4.3 Summary of Sampling Scenarios

Table 1 summarizes our investigations of sampling and reconstruction scenarios. This clearly shows the relations of Definition 1 to Defintions 2 and 3. That is, over-sampling is a scenario satisfying both perfect reconstruction and redundant sampling scenarios. Similarly, normal sampling satisfies both perfect reconstruction and non-redundant sampling.

Interestingly, we can see in Table 1 that there are two types of under-sampling scenarios. Under-sampling scenario of type 1 is conventional. On the other hand, that of type 2 has never been pointed out explicitly. We show in the next section by means of examples that under-sampling scenario of type 2 may appear in practice.

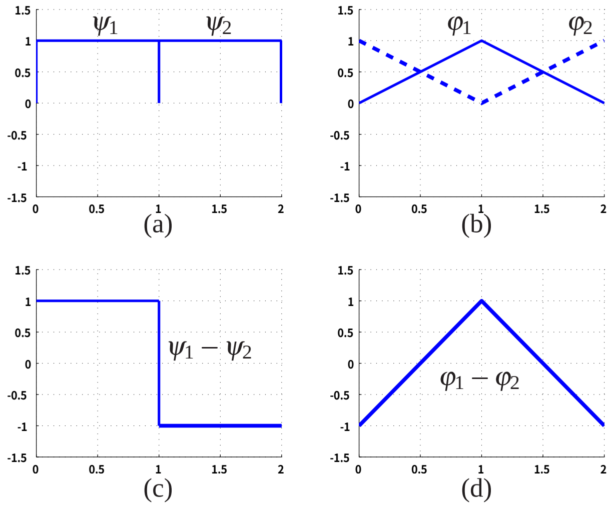

Figure 2: Under-sampling scenario of type 2. (a) Sampling functions. (b) Reconstruction functions. (c) An example of function in $V_s \cap V_r^{\perp}$. (d) An example of function in $V_r \cap V_s^{\perp}$.

5. EXAMPLES OF UNDER-SAMPLING OF TYPE 2

First, we show a toy example, which illustrates the geometric intuition behind our discussion. We use the B-splines $\beta^0(x)$ and $\beta^1(x)$ of degree 0 and 1 defined by

$$\beta^0(x) = \begin{cases} 1 & (0 \leq x < 1), \\ 0 & (x < 0, x \geq 1), \end{cases}$$

and

$$\beta^1(x) = (\beta^0 * \beta^0)(x),$$

respectively, where $*$ is the convolution operator. The details of the B-spline functions can be found in [11].

Let H be $L^2[0,K]$, where K is the number of the reconstruction functions. Functions f in H satisfy

$$\int_0^K |f(x)|^2 dx < \infty,$$

and the corresponding inner product is

$$\langle f, g \rangle = \frac{1}{K} \int_0^K f(x)\overline{g(x)} dx.$$

We consider the case of $N = K$. Let $\{\psi_n\}_{n=1}^{N}$ and $\{\varphi_k\}_{k=1}^{K}$ be functions given by

$$\psi_n(x) = \beta^0(x - n + 1),$$

$$\varphi_k(x) = \beta_K^1(x - k + 1),$$

respectively, where

$$\beta_K^1(x) = \sum_{k \in \mathbb{Z}} \beta^1(x - kK).$$

This corresponds to the periodized version of a system where the sampling is performed by integrating the signal over the sampling period ($T = 1$) and where the reconstruction is performed using piecewise linear splines.

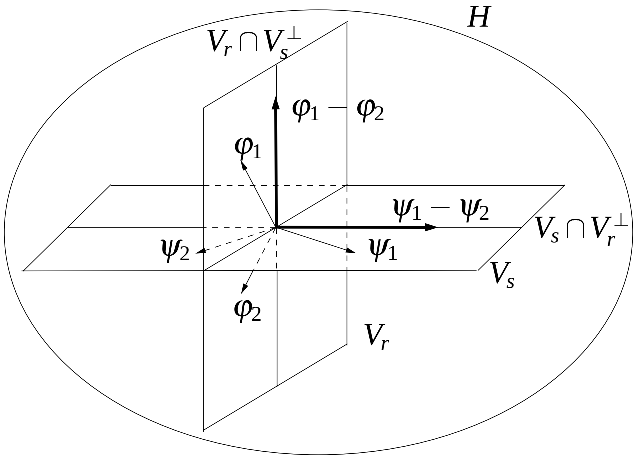

Figure 3: Geometric schema of the sampling and reconstruction scenarios in Section 5. We can see that $\psi_1 - \psi_2$ and $\varphi_1 - \varphi_2$ are perpendicular to V_r and V_s, respectively.

Now, in the case where $N = K = 2$, one can verify that $\tilde{f}$ in Eq. (8) with

$$\mathbf{c} = (1, -1)$$

belongs to $V_r \cap V_s^{\perp}$. That is, Eq. (13) is true. Further, it is easily shown that g in Eq. (20) with

$$\mathbf{a} = (1, -1)$$

belongs to $V_s \cap V_r^{\perp}$. That is, Eq. (19) holds. Hence, this scenario indeed corresponds to the under-sampling of type 2. These functions are shown in Fig. 2. Fig. 3 shows the geometric representation of the scenario. We can clearly see that $\psi_1 - \psi_2$ and $\varphi_1 - \varphi_2$ are perpendicular to V_r and V_s, respectively.

In a similar way, we show a more practical example. The sampling functions are the same as in the above example. The reconstruction functions are given by the B-spline of degree 3. That is, by letting

$$\beta_K^3(x) = \sum_{k \in \mathbb{Z}} \beta^3(x - kK),$$

the reconstruction function is given by

$$\varphi_k(x) = \beta_K^3(x - k + 1).$$

We assume that K is even, and $N = K$. In this case, similarly to the example above, $\tilde{f}$ in Eq. (8) with

$$\mathbf{c} = (1, -1, \ldots, 1, -1)$$

belongs to $V_r \cap V_s^{\perp}$. Further, g in Eq. (20) with

$$\mathbf{a} = (1, -1, \ldots, 1, -1)$$

belongs to $V_s \cap V_r^{\perp}$. Hence, this scenario also corresponds to under-sampling of type 2.

6. CONCLUSION

In this paper, we investigated over-sampling and under-sampling scenarios in the formulation of a generalized sampling model. We first reviewed the conventional definitions of over-sampling and under-sampling, and pointed out they consist of two conditions. To treat them separately, we introduced two concepts, perfect reconstruction and redundant sampling. We showed that these scenarios are geometrically characterized by means of sampling and reconstruction spaces. Then, we showed four possible scenarios. Especially, an interesting under-sampling scenario (called “of type 2”) appeared. We showed by a practical example that it may be encountered in real applications. The exploitation of its characteristics in practical problems is a promising issue. We will concentrate our future work on the way to deal with the situation where the measurements are corrupted by noise.

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